THE ARITHMETIC TEACHER

Volume IV Number 3

April 1957

Six Years of Research on Arithmetic Instruction: 1951-1956

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The improvement of instruction in arithmetic is, as in other areas of the curriculum, a function of numerous factors. Without question one of the most important of these is *research*. The amount, quality and significance of research activity; the extent to which research findings are reported and disseminated; the degree to which the implications of research are implemented in actual classroom situations—all these have a marked influence on the amount of instructional improvement we can hope for in the field of arithmetic.

Research Bibliographies

Research listings, bibliographies, summaries and the like covering elementary-school mathematics instruction serve an obviously useful purpose for both the research worker and the classroom teacher. Quite understandably, however, the same material may not meet the needs of each equally well. In any event we all have been helped by things such as the three-year critical summaries in the *Review of Educational Research*, the annual annotated bibliographies in the *Elementary School Journal*, and other listings and summaries which have appeared at irregular intervals as special monographs or in various journals such as The Arithmetic Teacher.

It is most unfortunate that the three-year critical summaries of research on arithmetic appearing in the *Review of Educational Research* were discontinued, at least temporarily, after the October 1951 issue. For this and other reasons there seemed to be value in preparing some form of listing covering research on elementary-school mathematics instruction specifically for the calendar years 1951 through 1956, even though several different types of summaries have been published within the past six years. The present material represents the result of that feeling.

Delimitation of Listing

At the outset is was decided to strive for a comprehensive rather than a selected listing of pertinent research and closely related literature reported during the six-year period, 1951–56. Thus it was necessary to limit the summary to *published* material and not even attempt to include unpublished theses, dissertations, etc. completed during the period in question.

A much more difficult decision centered around what should be included as "research," since this term has a somewhat variable breadth of connotation. It was decided rather arbitrarily that for purposes of this listing only the following should be included: (1) normative and

experimental studies which report specific data or findings on a problem associated with mathematics instruction in the elementary grades, either directly or by implication; and (2) annotated bibliographies, summaries, and more or less critical discussions which relate, in whole or major part, to such normative and experimental studies. It is hoped that the reader will not find too much fallibility of judgment on the writer's part in his compilation of the present listing on the basis of these criteria!

Nature of Annotations

The references included in this listing are numbered consecutively and arranged alphabetically by author. Each reference is followed by a brief one-sentence annotation which seeks to describe the entry more clearly than is possible by its title alone. Admittedly, this form of annotation leaves much to be desired for certain purposes. However, it should be adequate to give the reader a reasonably clear indication of the general nature of each reference so that he may select those he wishes for special reading and study from the standpoint of his own interests and purposes.

Concluding Observations

A section of concluding observations follows the annotated listing. Here the writer has attempted to summarize the entries on the basis of several categories and to make some relevant personal observations regarding the research published during the six-year period, 1951–56.

Research Bibliography

1. BENZ, HARRY E. "Two-Digit Divisors Ending in 4, 5, or 6." ARITHMETIC TEACHER 3: 187–91; November 1956.

Discusses aspects of the "one-rule" vs. "two-rule" controversy for estimating quotient digits when dividing by two-figure divisors, presenting and analyzing data regarding the occurrence of various types of examples with selected divisors.

2. BERNSTEIN, ALLEN. "A Study of Remedial Arithmetic Conducted with Ninth Grade Students." *School Science and Mathematics* 56: 25–31 and 429–37; January and June 1956.

Reports a two-year study involving diagnosis and remediation of arithmetic difficulties among students at the ninth-grade level.

3. BOUCHARD, JOHN B. "An Exploratory Investigation of the Effect of Certain Selected Factors upon Performance of Sixth-Grade Children in Arithmetic." *Journal of Experimental Education* 20: 105–12; September 1951.

Reports a study of the effects on sixth-graders' arithmetic test performance when information about selected factors relating to the testing situation is imparted to the children.

4. BROWNELL, WILLIAM A. "Arithmetical Readiness as a Practical Classroom Concept." *Elementary School Journal* 52: 15–22; September 1951.

Reports a study of the measured readiness of fifth-grade children for division by two-place divisors as determined by an appropriate readiness test administered at the point when classroom teachers were about to begin systematic instruction on the new topic.

5. ———. "The Effects of Practicing a Complex Arithmetical Skill upon Proficiency in Its Constituent Skills." *Journal of Educational Psychology* 44: 65–81; February 1953.

Reports an extension of the preceding study (No. 4 in this listing) based on a re-administration of the original readiness test following a period of systematic instruction in division with two-digit divisors.

6. BURCH, ROBERT L. "Formal Analysis as a Problem-Solving Procedure." *Journal of Education* 136: 44–47, 64; November 1953.

Reports an experimental study evaluating the effectiveness of a formal analysis

procedure in the solution of arithmetic problems at the fourth-, fifth- and sixth-grade levels.

7. ——— and Harold E. Moser. "The Teaching of Mathematics in Grades I through VIII." *Review of Educational Research* 21: 290–304; October 1951.

Summarizes and discusses research and related literature on the teaching of mathematics in the first eight grades reported during the preceding three-year period.

8. BUSWELL, GUY T. "Needed Research on Arithmetic." *The Teaching of Arithmetic.* Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. p. 282–97.

Presents summaries of research proposals submitted by 21 persons who had contributed previously, in one way or another, to the research literature on arithmetic instruction.

9. ———, with the cooperation of Bert Y. Kersh. *Patterns of Thinking in Solving Problems.* University of California Publications in Education, Vol. 12, No 2. Berkeley and Los Angeles: University of California Press, 1956. p. 63–148.

Reports a study of patterns of thinking used by high school and university students in solving problems expressed in arithmetical terms.

10. DAWSON, DAN T. "Number Grouping as a Function of Complexity." *Elementary School Journal* 54: 35–42; September 1953.

Reports a study, with selected supporting data collected at the first-grade level, which examined the hypothesis that the crucial factor in apprehending a number as a group is the degree of complexity of the perceptual field.

11. ——— and Arden K. Ruddell. "An Experimental Approach to the Division Idea." ARITHMETIC TEACHER 2: 6–9; February 1955.

Reports a study comparing a "subtraction" approach with a somewhat "more conventional" approach to aspects of instruction involving the division process with whole numbers at the fourth-grade level.

12. ———. "The Case for the Meaning Theory in Teaching Arithmetic." *Elementary School Journal* 55: 393–99; March 1955.

Summarizes the findings and implications of five "critical" and "significant" studies relating in one way or another to the so-called "meaning theory" of arithmetic instruction.

13. DUTTON, WILBUR H. "Attitudes of Junior High School Pupils toward Arithmetic." *School Review* 64: 18–22; January 1956.

Reports a study of attitudes of junior high school pupils toward arithmetic as measured by an attitude scale constructed by the author. (See No. 15 below in this listing.)

14. ———. "Attitudes of Prospective Teachers toward Arithmetic." *Elementary School Journal* 52: 84–90; October 1951.

Reports on the nature of attitudes toward arithmetic on the part of undergraduate teachers-in-training as revealed by their own written statements.

15. ———. "Measuring Attitudes toward Arithmetic." *Elementary School Journal* 55: 24–31; September 1954.

Reports data regarding the construction and use of an arithmetic attitude scale with prospective teachers, chiefly undergraduates, at the university level.

16. DYER, HENRY S., Robert Kalin and Frederic M. Lord. *Problems in Mathematical Education.* Princeton, N. J.: Educational Testing Service, 1956. 50 p.

Reports a survey of crucial current issues and problems in mathematics instruction and education, with suggestions for needed research and related activity.

17. EADS, LAURA K. "Ten Years of Meaningful Arithmetic in New York City." ARITHMETIC TEACHER 2: 142–47; December 1955.

Reports an "action research" program involving some 20,000 teachers and supervisors from the Kindergarten through Grade 6.

18. FEHR, HOWARD F., George McMeen and Max Sobel. "Using Hand-Operated Computing Machines in Learning Arithmetic." ARITHMETIC TEACHER 3: 145–50; October 1956.

Reports an experimental study evaluating the use of hand-operated computing machines by fifth-grade children during a period of one-half school year.

19. FLOURNOY, FRANCES. "The Controversy Regarding the Teaching of Higher-Decade Addition." ARITHMETIC TEACHER 3: 170–73, 176; October 1956.

Reports findings from a survey of children's textbook series and of professional books regarding the initiation and development of work with higher-decade addition, especially in Grade 3.

20. ———. "The Effectiveness of Instruction in Mental Arithmetic." *Elementary School Journal* 55: 148–53; November 1954.

Presents sixth-grade data from a study evaluating the effectiveness of a systematic mental-arithmetic program implemented at the intermediate-grades level.

21. GIBB, E. GLENADINE. "A Review of a Decade of Experimental Studies Which Compared Methods of Teaching Arithmetic." *Journal of Educational Research* 46: 603–08; April 1953.

Summarizes eight experimental studies on the basis of four relevant aspects of each.

22. ———. "A Selected Bibliography of Research in the Teaching of Arithmetic." ARITHMETIC TEACHER 1: 20–22; April 1954.

Provides annotations for 18 selected research studies on the teaching of arithmetic reported within the preceding ten-year period.

23. ———. "Children's Thinking in the Process of Subtraction." *Journal of Experimental Education* 25: 71–80; September 1956.

Reports a study of the thought processes used by second-grade children when solving problems involving three kinds of situations commonly associated with the subtraction process.

24. ———. "Take-Away Is Not Enough!" ARITHMETIC TEACHER 1: 7–10; April 1954.

Suggests implications for problem-solving in the primary grades based on reported findings from the author's study of "Children's Thinking in the Process of Subtraction." (See No. 23 above in the present listing.)

25. GLENNON, VINCENT J. and C. W. Hunnicutt. *What Does Research Say About Arithmetic?* Washington, D. C.: Association for Supervision and Curriculum Development, 1952, 45 p.

Summarizes research findings pertaining to 38 significant issues and questions relating to mathematics instruction in the elementary school.

26. GROSSNICKLE, FOSTER E. "The Training of Teachers of Arithmetic." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. p. 203–31.

Reports data relating to the academic and professional preparation of teachers of arithmetic, based on questionnaire returns from state teachers colleges and on a study of college catalogs and pertinent related literature.

27. GUNDERSON, AGNES G. "Arithmetic for Today's Six- and Seven-Year-Olds." ARITHMETIC TEACHER 2: 95–101; November 1955.

Reports selected findings from a study of thought patterns used by second-grade children when working with multiplication and division problem-situations. (See No. 28 below in the present listing.)

28. ———. "Thought-Patterns of Young Children in Learning Multiplication and Division." *Elementary School Journal* 55; 453–61; April 1955.

Reports a study of thought patterns used by a group of second-grade children when working with multiplication and division problem-situations.

29. HALL, JACK V. "Business Uses of Mental Arithmetic in Ellensburg, Washington." *Journal of Educational Research* 46: 365–69; January 1953.

Reports a study of the uses of mental arithmetic based on interviews with 126 business persons drawn from 62 different occupational fields.

30. ———. "Mental Arithmetic: Misunderstood Terms and Meanings." *Elementary School Journal* 54: 349–53; February 1954.

Reports a survey of terminologies used to indicate problems to be solved mentally in fifth- and sixth-grade arithmetic textbooks and in the professional literature.

31. HARTUNG, MAURICE L. "Selected References on Elementary-School Instruction: Arithmetic." *Elementary School Journal* 57: 161–62; December 1956. 56: 124–25; November 1955. 55: 162–64; November 1954. 54: 163–65; November 1953. 53: 163–65; November 1952. 52: 162–64; November 1951.

Each represents an annual annotated bibliography of selected references on arithmetic instruction in the elementary school, some of which relate to research studies.

32. HARVEY, LOIS FULCHER. "Improving Arithmetic Skills by Testing and Reteaching." *Elementary School Journal* 53: 402–09; March 1953.

Reports data from that phase of a sixth-grade program of diagnostic testing-reteaching-retesting which dealt with so-called "zero errors" in the multiplication of whole numbers.

33. HIGHTOWER, H. W. "Effect of Instructional Procedures on Achievement in Fundamental Operations in Arithmetic." *Educational Administration and Supervision* 40: 336–48; October 1954.

Summarizes selected research findings on three questions concerning the relative merits of different methods of adding and subtracting whole numbers.

34. HILL, EDWIN H. "Teachers! Two Kinds of Division." *Journal of Education* 137: 16–18; May 1955.

In connection with a discussion of "measurement" and "partitive" division, reports several findings from the author's unpublished dissertation, *A Study of Preferences and Performance of Third, Fourth, Fifth, and Sixth Grade Children on Two Types of Division Problems.*

35. HIRSCH, MARTIN. "Does Changing the Form of a Problem Affect Its Difficulty?" *High Points* 33: 19–25; December 1951.

Reports a study of the effect on the difficulty-level of three examples involving division with common fractions, taken from an eighth grade arithmetic test, when the form of stating the examples is changed.

36. HOEL, LESTA. "An Experiment in Clinical Procedures in Arithmetic." *Emerging Practices in Mathematics Education.* Twenty-Second Year book, National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1954. p. 222–32.

Describes the nature of a two-week summer clinic program in remedial arithmetic for children in Grades 6–8; reports some of the general results and summarizes several specific case histories.

37. ILG, FRANCES and Louise B. Ames. "Developmental Trends in Arithmetic." *Journal of Genetic Psychology* 79: 3–28; September 1951.

Reports observations and findings from a longitudinal study of some 30 young children in relation to their development of arithmetic concepts.

38. JOHNSON, J. T. "Decimal Versus Common Fractions." ARITHMETIC TEACHER 3: 201–03, 206; November 1956.

Reports data from two studies "conducted some time ago" with children at the fifth- and sixth-grade levels, comparing the use of common fractions with decimal fractions in addition and subtraction examples expressed only in common fractional form.

39. LAYTON, W. I. "Mathematical Training Prescribed by Teachers Colleges in the Preparation of Elementary

Teachers." *Mathematics Teacher* 44: 551–56; December 1951.

Summarizes aspects of mathematical training prescribed by teacher preparation institutions for elementary education majors, based on a study of 85 college catalogs.

40. MARTIN, WILLIAM E. "Quantitative Expression in Young Children." *Genetic Psychology Monographs* 44: 147–219; November 1951.

Reports findings from a developmental study of various aspects of quantitative expression on the part of young children.

41. MCLATCHY, JOSEPHINE H. "The Pre-School Child's Familiarity with Measurement." *Education* 71: 479–82; April 1951.

Reports findings from a study of children's familiarity with common measurement concepts and instruments, based on an inventory test used with three-, four- and five-year-olds.

42. MORTON, R. L. *Teaching Arithmetic.* "What Research Says to the Teacher," No. 2. Washington, D. C.: National Education Association, 1953. 33 p.

Interprets for the classroom teacher the implications of research on selected phases of arithmetic instruction from the author's characteristic point of view.

43. MULHOLLAND, VERNIE. "Fifth Grade Children Discover Fractions." *School Science and Mathematics* 54: 13–30; January 1954.

Reports a description and evaluation of the work of a fifth-grade class in developing fractional concepts and skills through a program emphasizing visual and manipulative materials.

44. NELSON, THEODORA. "Results of General Mathematics Tests." ARITHMETIC TEACHER 3: 21–26; February 1956.

Reports results from administering tests in general mathematics in connection with the Nebraska Inter-High School Contests over a five-year period.

45. ORLEANS, JACOB S. *The Understanding of Arithmetic Processes and Concepts Possessed by Teachers of Arithmetic.* Publication No. 12. New York: Office of Research and Evaluation, Division of Teacher Education, College of the City of New York, 1952. 59 p. (Mimeographed)

Reports data relating to the construction of a test of mathematical understandings and its administration to teachers-in-service.

46. ——— and J. L. Sperling. "Arithmetic Knowledge of Graduate Students." *Journal of Educational Research* 48: 177–86; November 1954.

Reports findings from a study of arithmetic difficulties and deficiencies evidenced by students working toward graduate degrees in education.

47. ——— and Edwin Wandt. "The Understanding of Arithmetic Possessed by Teachers." *Elementary School Journal* 53: 501–07; May 1953.

Presents an abbreviated report of selected portions of the data and findings appearing in the Orleans monograph cited previously. (No. 45 in this present listing.)

48. PETTY, OLAN. "Non-Pencil-and-Paper Solution of Problems." ARITHMETIC TEACHER 3: 229–35; December 1956.

Reports an experimental study of the effectiveness of a non-pencil-and-paper method for solving verbal arithmetic problems when used with children at the sixth-grade level.

49. PHILLIPS, CLARENCE. "Background and Mathematical Achievement of Elementary Education Students in Arithmetic for Teachers." *School Science and Mathematics* 53: 48–52; January 1953.

Reports a survey of aspects of mathematical background and competence on the part of university students, mostly seniors, enrolled in an arithmetic methods course.

50. PHILLIPS, JOHN L., Jr. "Perception in Number Skills—a Study in Tachistoscopic Training." *Journal of Educational Psychology* 45: 459–67; December 1954.

Reports a comparative study of two

tachistoscopic methods and a workbook method of practice in learning selected multiplication and division facts at the fourth-grade level.

51. Poffenberger, Thomas and Donald A. Norton. "Factors Determining Attitudes toward Arithmetic and Mathematics." Arithmetic Teacher 3: 113–16; April 1956.

Reports selected findings from a pilot study employing questionnaires and personal interviews with 16 university freshmen.

52. Rheins, Gladys B. and Joel L. Rheins. "A Comparison of Two Methods of Compound Subtraction." Arithmetic Teacher 2: 63–69; October 1955.

Reports a comparative study involving the decomposition and equal additions methods of compound subtraction with whole numbers as used by subjects at the eighth-grade level.

53. Sala, Vincent. "Industry's Use of Measures." *Education* 71: 487–90; April 1951.

Reports findings regarding the use of measures based on a survey of four industries, ranging from one requiring emphasis upon technical skill to one emphasizing predominantly manual work.

54. Schaaf, William L. "Arithmetic for Arithmetic Teachers." *School Science and Mathematics* 53: 537–43; October 1953.

Summarizes selected research and related literature on the mathematical preparation of prospective and experienced elementary teachers in connection with an outline of proposals for a content course in arithmetic for teachers.

55. Sherer, Lorraine. "Some Implications from Research in Arithmetic." *Childhood Education* 29: 320–24; March 1953.

Discusses certain implications of selected types of research studies relating to important instructional matters which primarily are noncomputational in nature.

56. Skinner, B. F. "The Science of Learning and the Art of Teaching." *Harvard Educational Review* 24: 86–97; Spring 1954.

Discusses an experimental application of the author's unique theory of learning to arithmetic instruction through use of a mechanical device providing controlled reinforcement.

57. Smith, Linda C. "Concept of Money via Experience." Arithmetic Teacher 2: 17–20; February 1955.

Reports aspects of a study based on use of an "experience unit" centering around U. S. money in a fourth-grade classroom situation.

58. Snader, Daniel. "Mathematical Background for Teachers of Arithmetic." Arithmetic Teacher 3: 59–65; March 1956.

Reports results from a questionnaire study of the opinions of "specialists" concerning the content to be included in appropriate undergraduate courses in mathematics *per se* for elementary education majors.

59. Spitzer, Herbert F. and Frances Flournoy. "Developing Facility in Solving Verbal Problems." Arithmetic Teacher 3: 177–82; November 1956.

Analyzes problem-solving procedures employed in four fifth-grade classroom situations; reports data relative to the types of special problem-solving techniques utilized in five current fifth-grade arithmetic textbooks.

60. Sueltz, Ben A. "Mathematical Understandings and Judgments Retained by College Freshmen." *Mathematics Teacher* 44: 13–19; January 1951.

Reports some of the results observed when tests of mathematical understandings and judgments designed for use with junior high school pupils were administered to college students just beginning their collegiate study of mathematics.

61. ———. "Twenty-five Questions on Arithmetic." Arithmetic Teacher 3: 250–51; December 1956.

Lists a group of questions directed to the author as Editor of The Arithmetic

TEACHER, many of which point toward needed research in the field of arithmetic instruction.

62. ULLRICH, ANNA. "Labeling Answers to Arithmetic Problems." ARITHMETIC TEACHER 2: 148–53; December 1955.

Reports a questionnaire study of teacher opinion and preference regarding the matter of labeling answers to arithmetic problems.

63. VAN ENGEN, HENRY and E. Glenadine Gibb. *General Mental Functions Associated with Division.* Educational Service Studies, No. 2. Cedar Falls, Iowa: Iowa State Teachers College, 1956. 181 p.

Reports a study comparing a "general-ideas" approach with a "unit skills" approach to aspects of instruction involving the division process with whole numbers at the fourth-grade level.

64. WEAVER, J. FRED. "A Crucial Problem in the Preparation of Elementary-School Teachers." *Elementary School Journal* 56: 255–61; February 1956.

Reports a study of mathematical understandings possessed by undergraduates majoring in elementary education and the growth in understandings resulting from relevant course work.

65. ______. "Differentiated Instruction in Arithmetic: An Overview and a Promising Trend." *Education* 74: 300–05; January 1954.

Summarizes selected research findings pertaining to individual differences among children in regard to arithmetic and related abilities.

66. ______. "Teacher Education in Arithmetic." *Review of Educational Research* 21: 317–20; October 1951.

Summarizes and discusses research and related literature on teacher education as it pertains to mathematics instruction in the elementary school during the preceding three-year period.

67. ______. "Whither Research on Compound Subtraction?" ARITHMETIC TEACHER 3: 17–20; February 1956.

Summarizes some of the research on compound subtraction as a basis for examining critically certain aspects of the study reported by Rheins and Rheins (No. 52 in this present listing); briefly reports new findings and suggests a direction for further research.

68. WILSON, GUY M. "Toward Perfect Scores in Arithmetic Fundamentals." ARITHMETIC TEACHER 1: 13–17; December 1954.

Summarizes data from studies directed by the author during the period 1928–42 pertaining to the goal of 100% accuracy, especially in relation to the addition of whole numbers.

69. ______. and Mabel Cassell. "A Research on Weights and Measures." *Journal of Educational Research* 46: 575–85; April 1953.

Reports a survey of elementary- and high-school pupils relative to their knowledge of commonly used weights and measures.

70. WRIGHTSTONE, J. WAYNE. "Constructing Tests of Mathematical Concepts for Young Children." ARITHMETIC TEACHER 3: 81–84; 108; April 1956.

Describes the construction, validation and determination of percentile norms of tests of mathematical concepts for use in Grades 1 and 2; also reports reliability data for the tests.

71. ______. "Influence of Research on Instruction in Arithmetic." *Mathematics Teacher* 45: 187–92; March 1952.

Draws upon 27 studies and summaries as the basis for discussing the influence of research on selected aspects of arithmetic instruction.

Caution in Interpretation

A final word of explanation regarding the preceding bibliography may be in order. Although it was the writer's intention to make this listing of research on arithmetic as complete as possible, some potential entries were omitted purposely. For instance, one study involving the addition process was not included because

the primary purpose of the investigation centered around the psychological concept of "set" rather than upon anything arithmetic. Another research report was not listed because it dealt primarily with the statistical techniques used in a study involving fractions and directed no real attention to the findings from the standpoint of arithmetic instruction. A few other omissions were made deliberately for similar or related reasons.* Beyond these situations, any errors of omission either were committed inadvertently or were due to the fallibility of the writer's judgment in applying the criteria mentioned at the outset of this report.

As the reader now moves along to some of the classifications and interpretations which follow, it is well to be fully aware of several things regarding the basic listing. First, in two instances a single bibliographic citation has been used to cover two or more reports from the professional literature. Specifically, citation No. 2 covers reports from two different issues of the stated periodical. Similarly, citation No. 31 covers reports from six different issues of the specified periodical.

Second, in several instances all or parts of the same basic research study appeared in more than one source. Each report, of course, is listed as a separate citation in the bibliography. In most instances it is quite evident when this condition prevails, however. For example, compare the following pairs of citations: Nos. 4 and 5, Nos. 23 and 24, Nos. 27 and 28, etc. In such cases it is generally true that the pairs of reports are not duplicates of each other. They are usually written from different points of view or highlight somewhat different emphases.

It is hoped that facts such as these will assist the reader in exercising caution in interpreting portions of the material which follows.

* In at least two instances the writer did not include reports of research studies which appeared in monographs or brochures published by business concerns having definite "vested interests" in the research projects.

Sources of Research Reports

In what professional sources were research reports on arithmetic instruction found most commonly? This question may be answered in terms of the following summary which indicates the number of citations in the basic bibliography which appeared in each of the specified publications.

ARITHMETIC TEACHER	21
Elementary School Journal	12
Journal of Educational Research	4
School Science and Mathematics	4
Education	3
Mathematics Teacher	3
Journal of Education	2
Journal of Educational Psychology	2
Journal of Experimental Education	2
Review of Educational Research	2
Other Professional Periodicals (1 report each)	7
Professional Yearbooks and Monographs	9

It is both interesting and significant to note the large number of research reports which appeared in THE ARITHMETIC TEACHER. This is especially worthy of note since this new periodical has been in existence for only the last three of the six years covered by the bibliography—appearing first as a quarterly journal and even now being published only six times a year. We have good cause to be pleased with the research emphasis found in THE ARITHMETIC TEACHER. Long may it continue!

Types of Research Reports

In general, what major types of research reports were published during the six-year period, 1951–56? The answer to this question has been sought in terms of a three-fold classification: (1) reports giving emphasis to data, findings, implications, and/or the like from *experimental* studies; (2) reports giving emphasis to data, findings, implications, and/or the like from *normative* or *status* studies; and (3) reports which are research summaries, cri-

tical discussions, annotated listings, etc., or which do not reasonably fall in either category (1) or (2).

The reader very well may differ from the writer in his judgment of the 71 citations in the basic listing in relation to the three categories above, and for understandable reasons. The classification of some citations is not a clear-cut matter and might be handled differently by different persons. In any event, the writer's interpretations are summarized in the tabulation below which indicates the number of citations classified in each of the three categories.

Reports of experimental studies	22
Reports of normative studies	28
Other types of reports	21

The reader may draw his own inferences or conclusions regarding the relative attention given to major types of research studies on the basis of this tabulation.

Areas of Research Emphasis

What aspects or phases of arithmetic instruction have received greatest research attention or emphasis? The summary below indicates two things in an attempt to answer this question: (1) the number of research reports in the basic bibliography devoted to the most frequently investigated instructional problems, and (2) specific reference to each citation according to its numerical order in the basic listing.

It should be noted that some citations are placed intentionally in more than one category. Although numerous studies in the basic bibliography deal with "instructional method" in one way or another, a special category has not been used for classification on this basis.

MATHEMATICAL BACKGROUND AND PREPARATION OF TEACHERS, PRE-SERVICE AND IN-SERVICE (See citation Nos. 26, 39, 45, 47, 49, 54, 58, 64, 66)	9
MATHEMATICAL BACKGROUND AND PREPARATION OF OTHER STUDENT GROUPS (See citation Nos. 44, 46, 60)	3
PROBLEM SOLVING AND PATTERNS OF THINKING (See citation Nos. 6, 9, 20, 23, 24, 27, 28, 48, 59, 62)	10
DIVISION WITH TWO-DIGIT DIVISORS (See citation Nos. 1, 4, 5)	3
SUBTRACTIVE APPROACH TO DIVISION (See citation Nos. 11, 63)	2
MENTAL ARITHMETIC (See citation Nos. 20, 29, 30, 48)	4
ATTITUDES TOWARD ARITHMETIC (See citation Nos. 13, 14, 15, 51)	4
INDIVIDUAL DIFFERENCES, DIAGNOSTIC-REMEDIAL PROGRAMS (See citation Nos. 2, 32, 36, 65)	4
MEASUREMENT CONCEPTS, ETC (See citation Nos. 41, 53, 69)	3
COMPOUND SUBTRACTION (See citation Nos. 52, 67)	2

Two things clearly stand out as commanding greatest research interest during the six-year period in terms of the number of reported investigations, etc.: (1) the mathematical background and preparation of teachers at the pre-service and in-service levels, and (2) aspects of problem solving, especially the study of thinking patterns used in various problem situations.

The writer personally is pleased to see that more than passing attention of a research nature has been given to "mental arithmetic" and to one of the so-called *intangibles* of instruction: "attitudes."

Sustained Research Effort

An important part of the research picture in any instructional field is to be observed in the number of persons giving evidence of a sustained interest and contribution in terms of reported investigations. In this connection it is good to see that some of the major contributors to our research literature in past years have continued to add reports of new investigations during the 1951–56 period.

As a corollary to this observation, we may ask: to what extent has there been a sustained research contribution on the part of some investigators during the six-year period presently being reviewed? In answer to this question we can point to the fact that *eight* persons have more than

one experimental or normative research report appearing among the citations in the basic bibliography. When we take into account that some investigators have devoted more than one report to a particular study, we reduce to 5 or 6 the number of persons who have reported two or more *different* experimental or normative investigations during the 1951–56 period. The reader may judge for himself the extent to which this reflects a desirable degree of sustained research interest and activity.

Significance of Research

As would be expected, the citations in the basic bibliography differ materially in their quality and significance as contributions to our professional literature. The writer is not so presumptive, though, as to attempt any form of multi-fold classification on such a basis.

Nevertheless, it seems to the writer that it would be valid to contend that there have been all too few truly major and significant research studies on arithmetic instruction reported during the six-year period, 1951–56. This is especially true if we think in terms of *experimental* studies which deal directly with fundamental aspects of the learning process. From this standpoint the amount of truly significant research reported during 1951–56 does not seem to measure up to that of some previous six-year periods.

In this connection it should be mentioned that, upon occasion, some potentially significant research studies lose much of their impact because of the more or less abbreviated form in which they are reported. It may be of some consequence to note in this regard that during the six-year period in question there was only *one* research study dealing directly with arithmetic instruction in the elementary grades that, to the best of the writer's knowledge, was published in the form of a comprehensive monograph. (See No. 63 in the present listing.)

A Look to the Future

The extent to which we may make truly significant advances in the area of arithmetic instruction during the years ahead will depend in large measure upon the nature of our research activity. The *amount* of reported research will not be as important as will be its emphases and its quality.

Numerous things are needed to improve the effectiveness and impact of our future research efforts. Among these only two will be mentioned as concluding thoughts for this summary and discussion of published research on arithmetic instruction during the six-year period, 1951–56. First, there is need for a more thoughtful identification of the truly crucial issues or problems we face in the teaching and learning of arithmetic in the elementary school. Second, there is need for more coordination of research attack from various sources upon these commonly recognized crucial aspects of arithmetic instruction.

The future can be a challenging and fruitful one from a research standpoint if we will accept the opportunity and make it such.

Editor's Note. We are all thankful to Dr. Weaver for his listing and brief annotation of research studies reported in journals and monographs during the past six calendar years. His work is doubly valuable through his classification and comment upon the types of research and the need for more basic studies.

Undoubtedly there are a number of unpublished researches that are filed in school offices and with committees on graduate degrees. The researchers who have produced this work should be encouraged to prepare reports for publication. The editor would very much appreciate having these. Many of them may be in the form of investigations and "projects" for the Ed.D. degree and hence may not be basically research but they do give information that may be useful to others and hence should be published.

It is the aim of The Arithmetic Teacher to continue to publish researches and investigations and to print each year a summary of the published research of the preceding year. Let us unearth more of the unpublished materials.

Estimating the Quotient in Division

A Critical Analysis of Research

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THERE ARE SEVERAL REASONS why the process of division in arithmetic is more difficult to learn than the other processes. One of these reasons stems from the fact that division is not a direct process. It is an inverse process in which it is frequently necessary to estimate in finding the quotient. In the past, two methods of estimating the quotient have been in common use. One of these is generally called the "apparent quotient" method and the other is called the "increase-by-one" method. There has been considerable research directed toward determining which of these methods is to be preferred for instructional purposes. The present article is not only a summary and critique of this research but also includes numerous comments upon aspects of the issue which have hitherto been largely neglected.

Connectionism Dominant

First, it should be noted that the research and discussion on the relative merits of methods of estimating the quotient has been dominated by the theory of learning known as *connectionism*. The influence of this approach to instructional problems is evident in the language used to explain the two major methods and even in the names customarily associated with them. This is illustrated in the following quotation from one of the early articles (2:299).*

"There are two well-known ways of estimating the quotient figure in long division. These are known as the "apparent method" and the "increase-by-one method." In the first of these methods the first figure of the divisor is used as a guide figure from which the quotient is estimated. In the second method the first figure of the divisor is used as a guide if the digit in the units' place is 5 or less; when the units' figure is 6 or more, the guide figure is increased by one. If the divisor is 54, the guide figure for estimating the quotient by both methods is 5. If 57 is the divisor, the guide figure in the apparent method is 5, but in the increase-by-one method the guide figure is 6."

In this quotation and throughout the literature terms such as "guide figure" and rules such as "increase-by-one" are typical of the connectionist approach. It is significant that one of the more recent articles (11:141) describes the methods in potentially more meaningful terms as follows. "Rounding the divisor to the next lower multiple of 10 as a method of estimating the quotient figure is commonly called the *apparent method.* Rounding the divisor to the next higher multiple of 10 is commonly called the *increase-by-one* method because the tens' figure of the divisor is increased by one." This description is potentially more meaningful because the concept of rounded numbers is general and widely applicable, and is readily understandable in terms of more basic concepts of the number system. It is important to note that this description does not limit the use of the "increase-by-one" method to cases where the second digit is 6, 7, 8, or 9, as the first description does.

* The notation (2:299) is used to refer to page 299 of the reference numbered 2 in the bibliography at the end of this article. Similar notations will be used for other references.

Various writers have also used terminology such as "Rule 1" or "Rule A" to refer to one of the methods, and "Rule 2" or "Rule B" to refer to the other. There are some advantages in the abstract nature of this nomenclature, but they do not seem to outweigh the loss in communicative power of more descriptive terms. Although reluctant to introduce still another nomenclature, the writer believes it is warranted in this situation. Hereafter the "apparent" method will be called the "round down" method.

Round Up or Down or Both?

It has never been customary to round the divisor up to the next multiple of ten unless the second digit is 6, 7, 8, or 9, but as Benz has noted in a recent article "it would be possible to teach pupils to work all division examples with the use of Rule B" (1:188). It is clear from the context that he means to round the divisor up to the next multiple of 10 even when the second digit is 1, 2, 3, 4, or 5. This later procedure will hereafter be referred to as the "round up" method.

Finally, since the "increase-by-one" method rounds *down* when the second digit of the divisor is small, and rounds *up* when it is large, it will be referred to as the "round both ways" method. Although not very elegant, this terminology is at least descriptive of what actually is done.

It should be observed that in estimating the quotient, it is almost universal practice to round the dividend *down*. Thus in the example $28\overline{)197}$, regardless of which method is used the dividend is treated as though it was 190.*

It would be convenient to have a simple and efficient method of finding the quotient in all examples, but no such rule exists. Under connectionist learning theory it is important to teach from the outset the procedures which characterize an efficient mature response. Modern instructional methods are not always aimed toward this sort of response in the early stages of learning. They do aim at the development of understanding of the arithmetic concepts and principles involved. Skilled performance, such as efficient and mature estimation of the quotient, is then sought as a later development in the process of learning.

Second, nearly all of the efforts to resolve the issue of which method is preferable have focussed upon a detailed analysis and comparison of the success of the methods when applied to what may be called an "example population." That is, a population of division examples is defined at the outset. Every example in the population is then attacked by each method, and a record is kept of the success or failure of the method. Ultimately, frequency counts are made of the success of the method, and in some cases, of the complications that arise when the method fails.

This sort of analysis of the content of the curriculum is characteristic of studies influenced by connectionist learning theory. Knight, who was one of the first investigators of the problem, made a comment on the usefulness of such studies (8:42).

> "Perhaps the only way to determine definitely the best method for the child to use in estimating the quotient figure is by actual experimentation on a large number of children. However, it is evident that such a research can be carried on satisfactorily only after the skills involved are understood at least in part. It is the purpose of this paper to present something of the nature and extent of the quotient difficulty.

At the time this was written thirty years ago teachers and authors of instructional materials were just becoming aware through research of the complexity of the learning situations faced by children. Although connectionist learning theory is now not as dominant as it was then, the data from such analyses of content are in some cases still useful. Regardless of the

* Grossnickle refers (5: 449) to the possibility of rounding the dividend up in certain special cases.

learning theory to be applied they provide leads to the design of experiments and the preparation of instructional materials. The studies of the relative success of various methods of estimating the quotient figure are cases in point.

Third, in commenting upon the work of several investigators Morton remarked (11:141) that "the results secured have differed because certain fundamental assumptions have differed." An analysis of these assumptions is instructive when examined in relation to the type of learning theory that seems to be applied. However, before undertaking this some points of general agreement among the various writers may be noted.

Range of Researches

In discussions of this problem only zero and the natural numbers indicated by the numerals 1, 2, 3, $\cdots$ are involved. The divisor, which will be indicated by d, is to be a "two-digit" numeral; more precisely, $10 \leqq d \leqq 99$. All writers except Osburn (13:441) exclude divisors which are multiples of 10; i.e., $d \neq 10y$, where $y = 1, 2, 3, \cdots, 9$. The quotient, which here will be indicated by q, is restricted to one digit,* but the range of q for some writers (Grossnickle, Upton, Osburn) is $0 \leqq q \leqq 9$, while for others (Knight-Jeep, Morton) it is $1 \leqq q \leqq 9$. This difference, which is not fundamental, is a consequence of varying assumptions about the domain of the dividend. All the writers agree that the dividend, which will here be denoted by D, should for any d not exceed $10d-1$, since otherwise the quotient q would equal or exceed 10. Certain writers (Grossnickle, Upton, Osburn) consider dividends down to zero ($0 \leqq D \leqq 10d-1$). Knight-Jeep and Benz exluded dividends smaller than the divisor ($d \leqq D \leqq 10d-1$), and Morton excluded the divisor also ($d+1 \leqq D \leqq 10d-1$).

Data on the dividend populations used by various investigators are summarized in Table 1. Although none of the investigators mentions the fact, it is easy to calculate the total population N by the use of arithmetic progression formulas. In case $0 \leqq D \leqq 10d-1$ the number of dividends for any particular d is $10d$. Then

$$N = \sum_{d=10}^{d=99} 10d = 49{,}050.$$

For multiples of 10 as divisors, $d = 10y$ where $1 \leqq y \leqq 9$. Thus for $d = 10, 20, 30, \cdots, 90$, the number of dividends is $\sum 10 \cdot 10y = 100 \sum y = 4500$. These dividends are commonly exluded, leaving a population of 44,550. Upton also excludes 810 cases for divisors 13, 14, $\cdots$, 18 which he proposes to treat by "trial and error" methods. Grossnickle calls these "division demons" and sometimes excludes them from portions of his discussion.

Several methods of calculating N are available when cases $D < d$ are to be excluded. The number of dividends for any d in this case is $9d$, and the total is $\sum 9d = 44{,}145$. The number in which the divisor is a multiple of 10 is $\sum 90y = 4050$, and the net population is $44{,}145 - 4050 = 40{,}095$. Morton also excluded 81 cases in which $D = d$, obtaining $N = 40{,}014$.

It will often be convenient to describe the divisor in terms of its tens' digit, which will be denoted by t, and by its units' digit, which will be denoted by u. Then $d = 10t + u$, where $1 \leqq t \leqq 9$ and $0 \leqq u \leqq 9$. When only cases $u = 6, 7, 8$, or 9 (or other selected values of u) are considered, methods similar to those above yield population figures for this sub-class of the total dividend population.

None of the investigators presents formulas for calculating the number of times a particular method of estimating

* This restriction involves no loss in generality. Knight pointed out (8: 43) that examples in which the quotient is two or more digits may be considered as several examples in each of which the quotient has only one digit. Grossnickle in two separate articles stated that his quotients could contain two digits, but it is clear from other evidence in these articles that his data are based on the range $0 \leqq q \leqq 9$ only.

TABLE 1

DOMAINS OF THE DIVIDEND AND DIVIDEND POPULATIONS CONSIDERED BY VARIOUS INVESTIGATORS

Investigator	Domain of Dividends, D	Dividend Population, N	Dividend Population $u>5$
Knight-Jeep	$d \leqq D \leqq 10d-1$	40,095	18,630
Grossnickle	$0 \leqq D \leqq 10d-1$	44,550	20,700
Upton	$0 \leqq D \leqq 10d-1$	43,740	18,090
Osburn	$0 \leqq D \leqq 10d-1$	49,050	20,700
Morton	$d+1 \leqq D \leqq 10d-1$	40,014	18,594
Karstens*	$2d \leqq D \leqq 10d-1$	35,640	—
Benz†	$d \leqq D \leqq 10d-1$	40,095	20,700

* Karstens considered only $u=5$.
† Benz considered only $u=4, 5, 6$.

the quotient will be successful on the first trial. It may be of interest, however, to know that correct estimates occur in systematic sequences, and that such formulas can be found. Thus under Morton's assumptions the number of "right" estimates R_1 by the "round up" method for any d is given by the formula

$$R_1=n_1\{2(10t+u)-(10-u)(1+n_1)\}/2,$$

where

$n_1=Q_1$ if $Q_1<9$, and $n_1=9$ if $Q_1 \geqq 9$,

and Q_1 is the quotient when the divisor $d=10t+u$ is divided by $10-u$. If, for example, $d=34$, then $10-u=6$, and $Q_1=5$. Since $5<9$, $n_1=5$, and $R_1=5\{2 \cdot 34-6 \cdot 6\}/2 =80$. If $d=76$, $10-u=4$ and $Q_1=19>9$; hence $n_1=9$, and $R_1=9\{2 \cdot 76-4 \cdot 19\}/2 =504$. If, as for $d=76$, $t+u \geqq 9$, then $n_1=9$, and the simpler formula $R_1=90t+54u-450$ may be used.

Similarly, the number of right estimates R_2 by the "round down" method for any d is given by the formula

$$R_2=n_2\{20t-(1+n_2)u\}/2-1,$$

where $n_2=Q_2$ if $Q_2 \leqq 9$, and $n_2=9$, if $Q_2>9$, and Q_2 is the quotient when $10t$ is divided by u. If, for example, $d=34$, then $Q_2=7$ and

$$R_2=7\{20 \cdot 3-8 \cdot 4\}/2-1=97.$$

If $d=76$, then $Q_2=11>9$, so $n_2=9$. Then

$$R_2=9\{20 \cdot 7-10 \cdot 6\}/2-1=359.$$

If, as for 76, $u \leqq t$, then the simpler formula $R_2=90t-45u-1$ may be used.

A summary of the number and per cent of estimates right on the first trial is given in Table 2. The results of Knight-Jeep are not included because they compared the "round both ways" method with the "round down" method for divisors

TABLE 2

NUMBER AND PER CENT OF ESTIMATES RIGHT ON FIRST TRIAL FOR VARIOUS INVESTIGATORS

Investigator	Round down		Round up		Round Both Ways		Dividend Population
	Number	Per cent	Number	Per cent	Number	Per cent	Number
Grossnickle	28,485	63.9	—	—	34,850	78.2	44,550
Upton	29,061	66.7	—	—	35,180	80.4	43,740
Osburn	*25,855**	*52.8*	*26,411*	*53.8*	*33,848*	*69.1*	49,050
Morton	21,264	53.1	24,515	61.3	*29,295*	*73.2*	40,014

* Italicized numerals were not explicitly reported by the investigator, but are derived from data he did report.

$u=6$, 7, 8, and 9. Their results if extended to their total dividend population N would be substantially the same as those of Morton. The results of Benz are not included because he presented data for only a sample of divisors among the subclass for which $u=4$, 5, or 6. Similarly, Karstens considered only the subclass $u=5$. Neither Grossnickle nor Upton give results for the "round up" method when $u<5$ under their assumptions. In fact, all the writers with the possible exception of Benz take it for granted no one would use the "round up" method for such samples.

An examination of the results as summarized in Table 2 shows that if success on the first trial is the criterion for selecting the method of estimating the quotient, then regardless of the varying assumptions involved in these investigations the "round both ways" method would be recommended. All the writers, however, discuss other considerations. For both Grossnickle and Osburn these other criteria sufficiently outweighed the one above that they recommended that the "round down" method be taught. Grossnickle, however, discussed (2:305) the advantages of a restricted form of "round both ways" in which the divisor is rounded up when $u=9$ and down for all other values of the units' digit. On the other hand, both Upton and Morton concluded that the other considerations did not sufficiently outweigh the advantages of rounding both ways, and they recommended that method.

This situation is analogous to an election in which the candidate who receives a majority of the votes is not necessarily declared elected. Instead, some authority weighs the various candidates' qualifications for office other than preferences of the electorate, and ultimately decides who shall hold office on the basis of all the evidence. It is highly significant that the theory of learning that is to apply is among the *other* considerations that seems to tip the scales one way or the other. The paragraphs that follow will consider in some detail the various assumptions that have been made in studying the problem of estimating quotients and will seek to interpret the results in terms of inferences about the learning theories that are implied.

Connectionism Plus

The comments fall into three main groups. The first group includes issues centering about the proper choice of the dividend population and how examples in which the dividend is less than the divisor are to be treated. The second group is concerned with a special situation that occurs when the trial quotient exceeds 9. A third group of comments includes issues centering about how many "rules" are desirable and the procedures to be followed when the estimated quotient is not the true quotient.

When the domain of the dividend population is $0 \leqq D \leqq 10d-1$, there are 4905 examples in which the dividend is a one- or two-digit number less than the divisor and the quotient is 0; examples are $26\overline{)17}$ and $26\overline{)24}$. If divisors which are multiples of 10 are excluded, there are 450 fewer, or 4455 such examples (Table 3).

It can be argued, as Grossnickle did (2:300), that for the sake of completeness these examples with dividends $0 \leqq D < d$ should be included in the example population. If this is granted, much depends upon how these examples are to be attacked by the pupil.

Although he does not explicitly say so, Grossnickle apparently counted the "apparent quotient" method as successful for 4050 of these examples. These were the cases in which the first digit of the dividend is less than the first digit of the divisor, as in the first example above. In the case of the 405 examples like $26\overline{)24}$, however, he apparently counted the method as unsuccessful. These counts suggest a routine or "mechanical" application of a rule—an approach from the stand-

TABLE 3

APPARENT QUOTIENT SUCCESS AS DETERMINED BY FOUR INVESTIGATORS IN CERTAIN SPECIAL CASES

Domain of D	Case	Example or Div. pop.	Grossnickle	Upton	Osburn	Morton
	Total	4905	considers	considers	considers	excludes
	$d=10y$	450	excludes	excludes	45	excludes
$0 \leqq D < d$	$T^* < t$	4050	4050 special rule	4050 special rule III (b)	405	excludes
	$T=t$ $U<u$	405	failure	405 special rule III (b)	failure	excludes
	$T=t$ $U=u$	81	81	81 special rule III (a)	81	excludes
$d \leqq D \leqq 10d-1$	$T=t$ $U>u$	324	324	324 special rule III (a)	324	324
	$H=t$	4050	3090 special rule	3090 special rule	failure	failure
	$d=10y$	4455	excludes	excludes	4050	excludes

* In this table T represents the tens' digit, U represents the units' digit, and H represents the hundreds' digit of the dividend.

point of connectionist learning theory.* It should be noted that Grossnickle did say that in all such examples "the result is quite evident," and thereby in effect assumed that the intelligent computer *observes* that the dividend is less than the divisor and does not apply the "apparent quotient" method mechanically. Otherwise, he might obtain a trial quotient of 8 in the example $26\overline{)17}$ by dividing 17 by 2, a procedure which would be legitimate in the example $26\overline{)170}$.

* Here and elsewhere in this article the writer does not intend to imply that any investigator would necessarily take today the same position that he did in a report written a quarter-century ago. The present discussion is, however, based on the *published research*, and is intended in part to show how differences in learning theories may affect decisions and results in any investigation.

Upton devised a special rule for cases of this kind. Under the heading Rule III (b) he says (15:263): "In cases like $17\overline{)13}$, $23\overline{)14}$, $29\overline{)17}$, etc. where the partial dividend is less than the divisor, it is *immediately seen* that the quotient is 0." It is interesting to observe that although his avowed purpose was "Making Long Division Automatic," Upton in a number of places discussed pupil responses that seem to depend upon meaning as well as specific habits and more or less "automatic" following of rules.

Osburn (13) applies the rules meticulously and finds the "apparent quotient" method successful in 405 out of 4455 examples in which the quotient is zero. The following are samples: $26\overline{)0}$, $26\overline{)1}$. If $d=26$ the method fails in the case of $26\overline{)2}$

and all dividends larger than 2 and less than 26; for other values of d it fails similarly except in 405 examples where D is a small number. Osburn includes multiples of 10 as divisors, so the method is successful in 45 examples such as $30\overline{)2}$ but fails for $30\overline{)3}$ and similar examples.

Morton's article was published much later (1947) when connectionist learning theory was less dominant. He excluded all such examples from the dividend population, and defended this decision in the following sentence (11:142): "No pupil who is ready for division by two-place numbers should find it difficult to estimate the quotient figure if the dividend is no larger than the divisor—the number of 38's in 38, or 29, or 16, or 5, or 0, for instance." This remark suggests that the pupil is expected to think about the relative size of the dividend and divisor and not apply a rule mechanically, and if so is characteristic of the recent approach to learning in arithmetic.

Morton excluded examples in which the dividend is equal to the divisor, and Upton used a special rule for dividends in the same decade as the divisor. As a result, Upton had fewer examples in which he applied the method, and had a total of 4860 examples which he tabulated separately as successes under his special Rule III. In effect, Upton relied upon a meaningful approach to these examples, but in keeping with the times he classified them under a formally stated set of rules.

When the "round up" method is considered for examples in which $D<d$ it appears that Grossnickle applied the method only to the 2070 examples possible when $u=6$, 7, 8, and 9. If used mechanically the method will then yield only 216 true quotients, but in order to secure his figure 34,850 for total success of the method Grossnickle seems to have given it credit for all of the 2070 possible examples. Osburn considered the entire dividend population ($u=0, 1, 2, \cdots, 9$) but his data do not permit a clear determination of how many successes he credited to the "round-up" method for $D<d$.* Upton used his special Rule III for these examples also, and Morton excluded them entirely.

This rather lengthy discussion of the complications that occur when $D<d$ illustrates the kind of issues that arise when connectionism as a learning theory is dominant. Such complications virtually disappear if reliance is placed upon pupil readiness and understanding. Modern instructional theory suggests that no pupil should be asked to undertake division by a two-digit number unless he first understands that a dividend smaller than the divisor cannot contain the divisor—or, more formally, that in such cases the quotient is 0. Thus Morton's position is defensible under modern learning theory although it probably is not if specific habits unsupported by understanding are viewed as the chief outcomes of instructional programs.

Another difficulty occurs when the dividend is a three-digit number in which the hundreds' digit is equal to the tens' digit of the divisor, but the tens' digit of the dividend is less than the tens' digit of the divisor; samples are $26\overline{)242}$ and $72\overline{)709}$. If the "apparent quotient" method and "guide figure" idea are applied mechanically in these examples, the result is not a true quotient. Thus Osburn states (13:446): "In the example $26\overline{)242}$ the apparent quotient is 1." Even if the more meaningful "round down" idea is used, so that the divisor 26 is thought of as rounded down to 20, the difficulty persists. In this case one may ask "how many 20's are there in 240?", or "how many 2's are in 24?" In both cases the result, 12, is still not the true quotient, 9, but it is much closer. Since true quotients can never exceed 9, it has been customary to tell pupils to "*try 9 and proceed*" in these examples. Thus a *special rule* is again invoked for this sub-class of examples.

* There is reason to believe but no evidence to show that the number was 540.

Grossnickle, to secure his figure of 28,485 (see Table 2), used this special rule and as a result included 3090 examples where its use gave the true quotient. Upton also used this rule. On the other hand, although Osburn prefers the "apparent quotient" method, he did not invoke it in these examples to build up his case. Morton, who prefers the "round both ways" method, also did not use the above rule and thereby avoided strengthening the case for the "round down" method.

In summary of this discussion of certain special difficulties, it should be noted that although varying assumptions tended to enhance the case for the "round down" method as far as the criterion of frequency of success is concerned, none of these clusters of assumptions resulted in total frequencies sufficient to carry the day over the other methods.

How Many Rules?

Another major group of issues centers around how many different methods or "rules" are desirable for instructional purposes. The general position is, of course, that the fewer rules there are, the better the learning situation is. The treatment of cases $D<d$ by a general principle is consistent with this position. The "round down" or "apparent quotient" method has sometimes also been called the "one rule" method because it is usually applied consistently regardless of the units' digit in the divisor. The "round both ways" method has been called the "two rule" method because one procedure is applied for certain values of u and a different procedure for other values of u. There has been much discussion based on the assumption that the learning of two methods or procedures places an added burden on the pupil and leads to confusion. Thus Osburn says (13:446): "Rule 1 has to be taught in any case, and the introduction of Rule 2 results in intolerable confusion. For the beginner and the slow learner, it is much better to stick to Rule 1 in all cases, with the instruction to try a quotient figure one less continually until the correct quotient figure is found."

Another important consideration is the correction that has to be made when a method fails on the first trial. When the "round down" method is used, if the trial quotient is not the true quotient, it is usually too large.* This means that the correction can be made by reducing the trial quotient. When the "round both ways" method is used, the trial quotient is sometimes larger and sometimes smaller than the true quotient. Grossnickle commented (3:442) as follows:

> "One of the most favorable features of the apparent method in estimating the quotient is that a uniform procedure is used for correcting an estimated figure which is not the true quotient. In this method the estimated figure is always made smaller. To determine whether the product of the divisor and the estimated quotient figure is less than the partial dividend, the pupil usually writes the product under the partial dividend. If the product is too large, this number is erased and the quotient figure is made one less. The product of the new quotient figure and the divisor is then compared with the partial dividend. This sequence is followed until the correct product is found."

Grossnickle outlines a variety of procedures for testing the accuracy of the estimated quotient, some of which are very complex and seldom used. Although Benz suggests that an eraser should not be regarded as an essential tool for long division, in practice it has been extensively used. The virtual necessity of many erasures and retrials is one of the disagreeable features of the "round down" method.

On the other hand, if the "round up" method is used the estimated quotient never exceeds 9 and if it is not the true quotient, is always smaller than the true quotient.* The need for a special rule

* Exceptions are examples such as $26\overline{)242}$ noted earlier in which the method is applied mechanically to obtain a trial quotient of 1 instead of 12, or 9 if a special rule is used.

* This statement assumes that cases $D<d$ are handled by "understanding" rather than by mechanically applying the rule.

when the estimate exceeds 9 is avoided. Moreover, the product of the trial quotient and the divisor is always smaller than the dividend, and the subtraction step can be done. Hence no erasures are necessary. If the remainder is not smaller than the divisor the pupil needs only to subtract additional multiples of the divisor from the remainder until it is smaller than the divisor, and then total the partial quotients he has found, as shown in the accompanying example (a). Although the amount of work as exhibited appears greater than in the conventional algorism, this is only because it is all shown. It is actually less. In the conventional form (b) the first trial quotient of 5, the product 235, and, if it has been calculated, the remainder 51 would usually all be erased. The next trial yields a quotient of 6, a product of 282, and a remainder of 4. This result is, of course, correct, and looks briefer because the work of the first trial was erased. The necessity of erasing, which is a natural accompaniment of the "round down" and "round both ways" method, is largely eliminated when the "round up" method is used. Moreover, recent research by VanEngen and Gibb (16) supports the assertion that the algorism described above is easier to understand than the traditional one. This is clearly a very important consideration when modern learning theory is considered.

(a)		(b)
47)286	5	47)286(6
235		282
51	1	4
47		
4	6	

In emphasizing the "one rule" aspect of the "round down" method, most writers have ignored the various special rules that are commonly attached to it.* Thus the instruction to "call the quotient 9 if the apparent quotient exceeds 9" is not only an auxiliary rule but can also be confusing to pupils. The "round down" method is far from infallible, as examples cited previously have shown. When these various special rules are taken into consideration, it becomes clear that the "round up" method comes much closer to being a "one rule" method than the others do. The force of tradition, however, seems to have inhibited any more than a superficial examination of its advantages.

When the number of corrections necessary is considered, the "round both ways" method involves the fewest and the "round down" method the most. Under Morton's assumptions the "round down" method is successful on the first trial in 53.1 per cent of the examples; the results for the "round up" and "round both ways" methods are 63.1 per cent and 73.2 per cent, respectively.

Osburn published data (14) which show that the "round down" method is successful in only 6855, or 33.2 per cent of the 20,700 examples with the units' digit greater than 5 that he included. The "round up" method succeeds with 14,858 or 71.7 per cent of them. These figures based on his data are comparable to those of Morton (6649 examples, or 35.8 per cent, and 14,680, or 79.0 per cent, based on 18,594 examples) and Upton, but Osburn himself did not comment on this. Instead, he criticized the work of Morton and Upton by asserting that they had

* For example, in a recent article Moser seems to have overstated the situation when he wrote (12: 519): "In the one-rule method we state a procedure which *always* works. When applied to any conceivable example, it will yield the correct quotient figure." The article by Moser is similar in some respects to the present discussion, but Moser does not emphasize the role of learning theory as the writer seeks to do. For so recent an article, his description of the details of procedure in the methods was put in terms that are remarkably similar to those used by writers who take a connectionist approach. Much of the "complexity" he ascribes to the "round up" method may be largely a consequence of this way of looking at it.

ignored 20,700 items of data for divisors $u=6$, 7, 8, and 9. It is more precise to say that these investigators by deliberate choice considered only 18,594 and 18,090, respectively, of these items, but did not break them down into sub-classes in the way Osburn thought they should. Osburn's analysis led him to the conclusion that the "round up" method for divisors $u=6$, 7, 8, and 9 is *helpful* in only 47 per cent of the 20,700 examples he included.

The requirement that a method be "helpful" in the sense of Osborn is more stringent than that it be merely successful, since with some examples both methods succeed on the first trial, or both methods fail. Moreover, it assumes that one method is taken as basic and the other is used as an auxiliary in the case of certain divisors. "Helpful" means that the method works when the basic method fails. Osburn assumed the "round down" method was basic and considered the "round up" method auxiliary. If, however, one assumes the "round up" method is basic and the "round down" is auxiliary, the situation is reversed. In this case, the same line of reasoning would show the "round down" or "apparent quotient" method is "helpful" in only 40.6 per cent of the 28,350 examples with divisors $u=0$, 1, 2, 3, 4, 5 considered by Osburn.

One type of comparison that has not hitherto been made involves comparing the success of the "round up" method for a divisor whose units' digit is u with the success of the "round down" method for the divisor in the same decade whose units' digit is $10-u$. For example, the "round up" method applied with the divisor 52 yields 144 true quotients on the first trial. The "round down" method applied with the divisor 58 yields 131 true quotients. The difference in this case is 13 examples in favor of the "round up" method. In fact, the "round up" method with Morton's assumptions produces more true quotients in every such comparison with larger differences as both tens' digit and units' digit increase. The sum of these differences is 3,251, the total number of examples by which the "round up" method exceeds the "round down" method for all divisors.

New Experimental Design Needed

There is one other question that may appropriately be considered here. Suppose these issues are to be studied by experimentation with children. How should such studies be designed? An early effort along these lines by Grossnickle (6) was indeterminate since he found no significant difference between the mean scores on a final criterion test taken by two groups of pupils. One group had been taught the "round down" method and the other had been taught the "round both ways" method for a period of seventy-six days. Although Grossnickle "equated" his groups and took steps in other ways to control his experiment, his design, his achievement tests, and his statistical tests of significance lack validity when judged by modern standards.

This is not the place to discuss the sort of experimental design which might help to settle these questions. It must suffice to say that the design must be rather sophisticated. Obviously sampling of the example population is necessary both for instructional purposes and criterion tests. It is not immediately evident how bias in these samples may be avoided. Difficulties mount when samples of pupils, of schools, and of teachers are to be chosen. Long division is not a simple process, and as Brownell has shown, the past experience and achievement of pupils with the necessary sub-abilities is far from uniform.

When all of these factors are taken into account, it becomes rather difficult to show statistically that one of these methods produces results that are superior to the others. One begins to wonder whether the problem is sufficiently important to warrant the complexity of the experimental machinery and the large amount

of work required in trying to reach a decision by this method. Perhaps the existing data together with the principles of learning theory accepted are sufficient to guide the curriculum maker to an intelligent decision in this case.

Summary

By way of summary, it may be noted that the "round up" method has not been taught until recently. This is surprising since: (1) it is successful on the first trial more frequently than the "round down" method; (2) it is a "one rule" method in which corrections if necessary are always upward; hence (3) no erasures are necessary when corrections are needed but work can continue; and (4) it avoids entirely the troublesome situations in which the trial quotient is larger than 9. For small values of u ($u=1$, 2, 3, and perhaps 4) several corrections may be needed in any one example, just as several are often needed for large values of u ($u=6$, 7, 8, 9) when the "round down" method is used.

When the "round up" method is compared with the "round both ways" it appears that: (1) the "round both ways" method is more frequently successful on the first trial; but (2) the "round both ways" method is a "two rule" method in which corrections if needed are either up or down; hence (3) erasures are sometimes needed. Both methods avoid the troublesome situations in which the trial quotient exceeds 9. Thus a choice between these methods depends very largely upon whether one believes that the greater complexity of the "round both ways" method is outweighed by the savings achieved through getting more estimates correct on the first trial.

It is surprising that those who favor the "round both ways" method have not urged its use when the units' digit is 5. Karstens alone has taken a firm stand in favor of this (7). Under Morton's assumption it yields the correct quotient for 58.5 per cent of the examples, whereas the "round down" method is successful for only 51.0 per cent. This suggests that practice has been determined here as elsewhere very largely by tradition rather than a rational basis.

The writer considers that the advantages of obtaining a trial quotient that is smaller than the true quotient and the relative simplicity of a "one rule" procedure outweigh gains in success on the first trial, and for at least the early stages of instruction therefore favors the "round up" method. Later, the pupils may be encouraged to "round down" when the units' digit is small, but in full awareness of the difficulties that may arise through an over-estimate.

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EDITOR'S NOTE. Professor Hartung has summarized very well the research and arguments for "rounding up," "rounding down," and "rounding both ways" of divisors in the operation of division. Division has been the topic of a great deal of writing and research because it is not a simple thing to learn. On the one hand, the situations that require the division process are of several types which are not always easily apparent as division problems. On the other hand, the process or operation is not only complex but is one in which a high order of thinking may be required. To lower the level of thinking many people have introduced special rules and procedures such as "round up," "use a table of multiples of the divisor," "use repeated subtraction," etc. Historically, our common algorism of division was developed by adults for adult use. It looks neat on paper but it is not easy to use.

It might be interesting to direct some research toward finding how "intelligent adults" perform division. Do they use the method they were taught in school or have they developed some useful supplementary techniques? How many adults who once knew how to divide are now almost totally at a loss with an exercise that has two or more figures in the divisor?

An assumption of previous researchs implies that all pupils should learn substantially the same method. This may not be valid. Perhaps we should have a method for average and better pupils and another for slower pupils. If we truly believe in "discovery" and "insight" and "meaning" and "understanding" we may have several methods in use in the same room. Shall we aim finally for as many pupils as possible to learn conventional division by some method of finding the correct quotient figure? Why not use Hartung's method (a) as illustrated by $286 \div 47$? Should all pupils try to learn to divide by more than one-digit divisors?

Keep Score on the Abacus

In many games found in the elementary classroom, there is the need for keeping score. Each one of us, regardless of age, likes to see the "little marks" showing to tell how many times we won the game or how many times our team scored.

A practical way of keeping score can be found in the use of an "open end" abacus. Each time "Johnny" wins he places a counter on the units' stick of the abacus. Then when he has won ten times, the ten counters are removed from the units' stick and one counter placed on the tens' stick of the abacus; and so on depending on the number of times it is necessary to group to higher values.

"Johnny" has not only kept scores, but in so doing has used his knowledge of grouping on the abacus.

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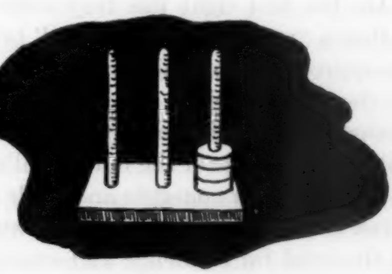
San Diego State College



Open-end Abacus

Arithmetic at the Primary Level

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Prin., Thomas N. Hart School, Boston

AT THE CURRICULUM CENTER for Arithmetic for the City of Boston for several years a series of classes has carried on some rather interesting experimentation in the field of elementary arithmetic. An attempt has been made to develop in the children an appreciation of the structure of our decimal system, an insight into the number relationships involved in this system, and the ability to think creatively with numbers.

To accomplish these outcomes we have relied for the most part upon the application of the psychological principles of learning associated with the newer theories of teaching arithmetic, upon the use wherever possible of manipulative materials for a clearer presentation of numerical concepts and processes, and upon the patience and kindness of fine teachers who in addition to developing an understanding of basic arithmetic concepts, endeavor also to awaken in the children that intangible concomitant of inspired teaching, a genuine love of arithmetic.

On the first sight this may seem to be rather a "reaching for the stars" but with an appreciation of child growth, especially in the area which is concerned with the various levels of maturity of thinking, with a knowledge of the individual's needs in basic number concepts and skills at any given time, and with a constant awareness of the need for challenge and experienced success, the outcomes we seek are not at all impossible of attainment to a great degree.

A major problem has been the sustaining of a steady growth in the child's power to think quantitatively. To accomplish this, number situations must be devised or must occur and be recognized in the day to day experiences of childhood whose understanding calls for the extension of previously mastered concepts and their application. This need poses a challenge to the teacher's creativeness and ingenuity. How can the new concept be presented so that from a recognition of its elements a new relationship may be discovered by the child? The answer to this question will determine to a great degree whether the teacher's guidance of the child's learning activities in arithmetic will be successful, for those concepts discovered by the child himself, and understood, will develop an insight into number relationships which is the basis of true perspective when the child progresses to more difficult and more complex problems. Perhaps a word or two here concerning insight might be worth while, for an understanding of this most desirable outcome of effective guidance in arithmetic learning will assist the teacher greatly in determining successive levels of maturity in the growth of the child's power to think. Webster defines insight as "Keen discernment or understanding; intuition; immediate cognition." As applied to arithmetic, this keen discernment, this intuitive process is the end product of previously acquired masteries. It has within it the ability to recognize instantly the component parts of a quantitative relationship and to assess the type of relationship and its implications. Hence a major element of insight is analysis and we attempt to attain these outcomes of insight and growth in the power to think in our third grade

activities in the field of multiplication by using an analytic approach to the mastery of multiplication facts.

Thus in our Primary program we endeavor to challenge each child to grow in power to think to the limit of his individual capacity and to become as efficient as possible in formulating generalizations. We encourage the independent discovery of number relationships. We seek to develop a true understanding of the basic concepts usually taught in the primary grades. When the foregoing aims have been accomplished as far as possible, we seek the achievement of mastery by a variety of meaningful drill and practice procedures which fix the skill of automatic response and develop ease and accuracy in the simple computations required in these grades.

The above is a statement of the philosophy which guides our thinking at the primary level. How can such a philosophy be implemented? How will such theory be put into practice? Probably the best procedure for providing answers to these questions would be to take a typical example from each of the three primary grade levels and identify our philosophy as we discuss technique. Thus the following example for Grade I.

I. Use of Structured Groups to Teach the Number Facts of Addition and Subtraction in the First Decade

The structured groups referred to here are the domino arrangements from 1 to 10.

A. MATERIALS NEEDED: A flannel board and circle cut-outs, each side of the circle having a distinctive color. Any colors may be used in place of the black and white combination used here. A piece of cardboard about 10×12.

B. We shall suppose that the class is ready for this phase of the work and can enumerate quantities at least to 10. We shall suppose further that the class has mastered the recognition of quantity by structure for the numbers 1, 2, 3 and 4 using these patterns (Fig. 1). We are about to introduce the number 5. Arrange 5 circles on

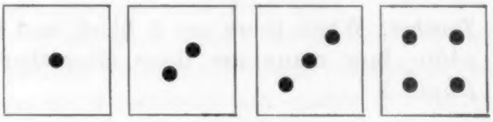


FIG. 1

flannel board as a domino pattern (Fig. 2) all the circles showing the same color—(black). The lesson will go somewhat along the following lines.

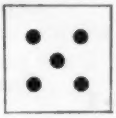


FIG. 2

Teacher: How many black circles do you see? *Pupil:* 5.

Teacher: How do you know there are 5?

Pupil: I counted them; (or) I see 4+1 more; (or) any response that satisfactorily explains why there are 5 black circles in this structured group. Now various class members will come to the flannel board and count the black circles. All are satisfied that there are 5 circles in the pattern. Now follows a brief review of the patterns from 1 to 5 using a tachistoscopic technique. An ordinary piece of cardboard may be used for this. With the cardboard covering the circles, the teacher makes any pattern from 1 to 5 and then with the words READY! WATCH! exposes the pattern briefly and again covers it. This technique is based upon the use of total form perception to associate a quantity with a particular pattern. Over a period of time it will also develop a skill in seeing. Since this technique also has the elements of a game, the child's interest is readily held. Now let us suppose that most of the children can recognize 5 by its structure in a domino pattern. Again the teacher places 5 black circles on the flannel board.

Teacher: How many black circles are there? *Pupil:* 5.

Teacher: How many white circles are there? *Pupil:* 0.

Teacher: When there are 5 black and 0 white, how many are there altogether? *Pupil:* 5.

Teacher: I am going to change this 5 in some way. Who can tell me how I change it? The teacher then covers the 5 with the cardboard and turns the middle circle over showing the white side. This leaves the basic 4 pattern unchanged.

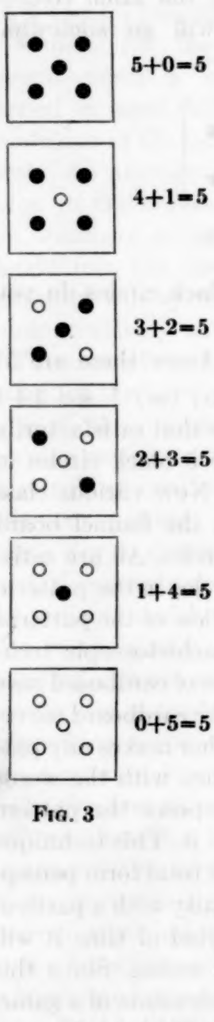


FIG. 3

Then the teacher, again using READY! WATCH! briefly exposes the pattern.

Teacher: What did you see?

Pupil: I saw 4 blacks and 1 white.

Teacher: How many did you see altogether?

Pupil: 5.

In this manner the teacher changes the color of the pattern to the following combinations remembering wherever possible to arrange the colors so that a basic pattern may be seen (Fig. 3).

In this way, all possible combinations of 5 are shown without breaking the overall pattern of 5 which is always seen though the combinations change. The combinations in seven are here also shown (Fig. 4).

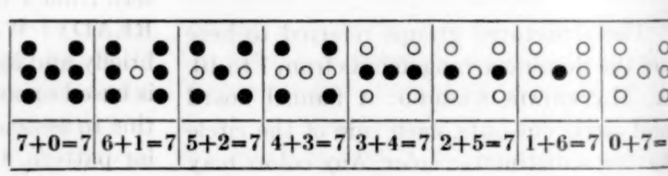
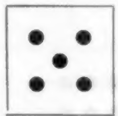


FIG. 4

With minor changes the patterns for 6, 8, 9 and 10 may be arranged so that basic patterns are combined within the overall structure. A similar grouping technique can bring out the subtraction facts of the first decade which can be learned along with the addition facts.

For subtraction, the teacher first arranges the pattern concealing it behind the cardboard. *Teacher:* I am going to show you a certain number. Then I am going to change it. I want you to tell me

how I change it. Now the teacher briefly uncovers the pattern (Fig. 5). Then cover-



ing the pattern takes the middle circle

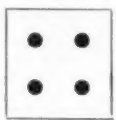


Fig. 5

away and uncovers the new pattern briefly. *Teacher:* Who can tell what I did? *Pupil:* You started with 5 and you took one away, so now there are 4.

If the pupil knows the structure of the pattern through total form perception, his frame of reference will enable him to recreate any part of the whole form which has been altered. Thus when a structural pattern has been altered as in the above subtraction technique, the pupil will readily recognize what has been changed. He also gets an early introduction to "taking-away" because the teacher actually takes the circles off the flannel board. The series for 5 would be as follows: (2 views) (Fig. 6).

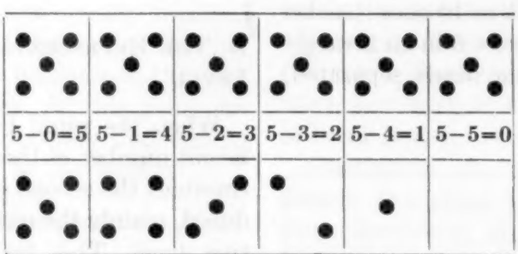


Fig. 6

II. *A Selected Grade II Technique in Subtraction*

An illustration of the three levels of learning involved in teaching with understanding the algorism:

$$\begin{array}{r} 42 \\ -17 \\ \hline \end{array}$$

A. First Level—the Manipulative

Materials needed: A 10 tens number frame.

Readiness: An understanding of the composition of number and a mastery of the subtraction facts.

A pupil is first asked to make the number 42 on the counting frame. He moves out 4 rows of ten and on the fifth row, 2 ones (Fig. 7).

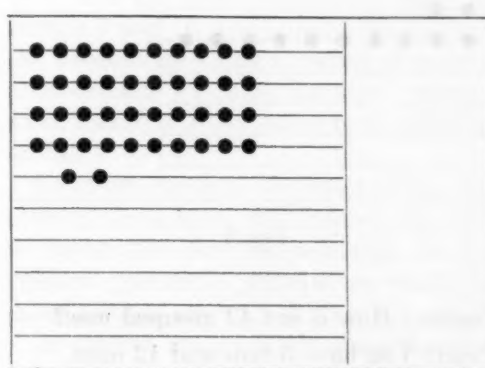


Fig. 7

Teacher: Now take away 17. At first trial subtraction may go something like this.

Pupil: First I'll take away 10, then I'll take away 2 ones and 5 more ones to make 7. This leaves me 2 tens and 5 ones which make 25.

Teacher: Now let us see what you did exactly. First you took away a ten then you took away the ones. Now let us try the same example again only this time let us take away the ones first. Can I take 7 ones from the 2 ones?

Pupil: No you can't.

Teacher: What must I do then to take away 7 ones?

Pupil: You'll have to get 5 ones from 1 of the tens.

Teacher: Then you mean we will have to change 1 of those tens into ones, so we can take 5 ones.

Pupil: That's right.

Teacher: Well then, let us take away the fourth ten (pushing 10 out of sight) and let us bring the 10 back as 10 ones (teacher brings out the ten on row 6 in an arrangement of ones:—the ten beads separated) (Fig. 8).

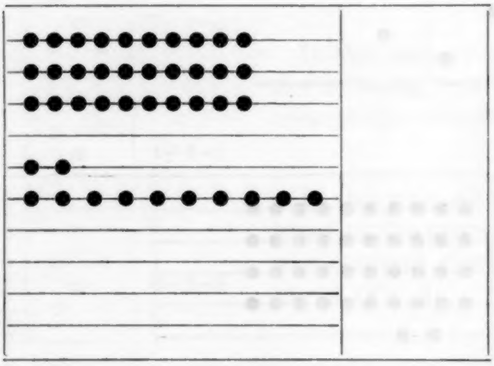


FIG. 8

Teacher: How is my 42 grouped now?

Pupil: You have 3 tens and 12 ones.

Teacher: Have I more or less than the number that I started with?

Pupil: Neither. You have the same amount.

Teacher: Well then, may I take away 7 ones now?

Pupil: Yes. Take away 2 ones and 5 more ones which leaves 5 ones. Now take away 1 ten from the 3 tens. This leaves 2 tens so my answer is 25. (Fig. 9).

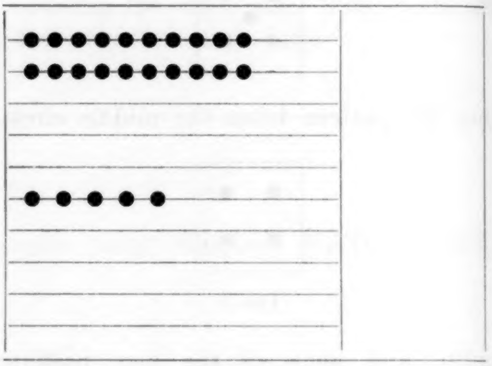


FIG. 9

B. THE REPRESENTATIVE OR RECORDING LEVEL

When the pupil has performed a sufficient number of these manipulative subtractions the second level should be introduced, mainly the recording or representative level. This technique teaches the child to record the concept or technique which he has discovered.

Teacher: Who can draw a picture of 42 on the chalkboard just as we made it on our number frame showing how many tens and ones there are in 42? (Fig. 10).

Teacher: We wish to take 17 from the 42. Can we take 7 ones from 2 ones?

Pupil: No. We have to take a ten and change it to 10 ones.

Teacher: How is our 42 grouped now?

Pupil: 3 tens and 12 ones (Fig. 11).

Teacher: Can we take the 7 ones away now?

Pupil: Yes. It leaves 5 ones.

Teacher: What else must we do to finish the subtraction?

Pupil: We must now take 1 ten from the 3 tens. This leaves 2 tens and my answer is 25 (Fig. 12).

	Tens	Ones
	10	1
	10	1
	10	
	10	
−	1	7

FIG. 10

	Tens	Ones
	10	1 1 1
	10	1 1 1
	10	1 1
		1 1
		1 1
−	1	7

FIG. 11

	Tens	Ones
	10	1
		1
	10	1
		1
		1
−	1	7
	2	5

FIG. 12

C. THE ALGORISM

When a sufficient number of subtractions have been worked on the manipulative and representative levels so that the process is thoroughly understood then the algorism is introduced. At this point the algorism will have meaning and should present no difficulty to the learner.

III. An Analytic Approach to the Multiplication Facts

Since the discovery and understanding of the number relationships involved in multiplication are basic to the growth of the individual's power to compute, a technique which develops the ability to analyze a number in the light of the relationships which it contains can be very useful in developing an insight into the structure of the decimal system and in particular into the structure of the first hundred. By this technique beginning with the number 1 and taking each succeeding number on a bead frame many useful relationships are discovered by the pupil. In the first decade for example the pupil learns to distinguish odd and even numbers; he discovers the multiplication facts involved in the decade; he discovers the principle of commutation; he learns that some numbers are prime and that these prime numbers are the building blocks used to make other numbers.

Encouraged to talk out the number situation as he manipulates the counting beads, the pupil will arrive at definitions which he truly understands. For example, one pupil decided, "A prime number is when you have to use the whole number or break it up into ones."

Continuing on through the second decade the pupil extends the relationships discovered in the first decade and finds some new relationships. He learns an odd number check for numbers divisible by 3 or 5 and later extends the check to include 7 and 11. As he progresses through the decades he keeps a record of certain number facts which he discovers, namely those found in the so-called multiplication tables.

From time to time also, after he has discovered these facts, drill periods are held so that automatic response may be developed. Desirable plateaus on which to pause for drill are 3×3, 4×4, 5×5, etc.

At 20 the manipulative stage for discovery may be abandoned for at this point the pupil will have sufficient power to proceed by analysis to determine the factors involved in each succeeding number. When the pupil has reduced a number to its prime components, he reassembles them to discover sets of factors. Having found the factors, he verifies them by using the counting frame. This he does as

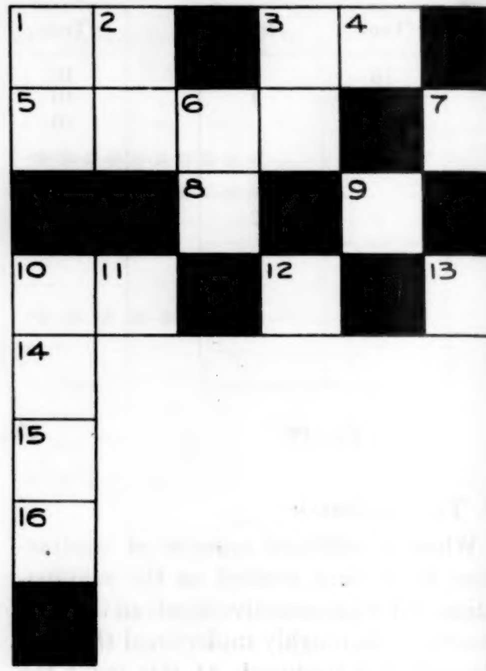
a check on his analysis. Using this analytic technique each child can progress according to his own ability and the pupils find the work challenging and interesting. A typical analysis of 36 would be as follows:

36	*Reasoning*	
$18\times2=36$	An even number divisible by 2	
	30 has	15 twos
	6 has	3 twos
	—	—
	36	18
$(9\times2)\times2=36$	18 has	9 twos
Prime line		
$(3\times3)\times2\times2=36$	9 has	3 threes
	All my factors are prime	
$3\times12=36$	Now to build the factors of 36.	
$9\times\ 4=36$		
$6\times\ 6=36$		
$18\times\ 2=36$		

Having discovered these number relationships in 36, the pupil verifies them at the bead frame and enters in his notebook, 9×4, 6×6, and the commutation 4×9.

EDITOR'S NOTE. Mr. Maloney has given us the method developed in the Boston Curriculum Center for Arithmetic. It is a method that uses understanding based upon analysis of numbers and the number system through first developing numbers objectively with visual and manipulative materials. It should be noted that the procedure and sequence is carefully designed so that there is a planned growth in understanding. Insight is nurtured, it is not assumed. New relationships are discovered by the child. He is not only permitted but encouraged to take "next steps" by himself. This challenge and active participation in learning helps to develop a genuine interest in learning arithmetic. But learning and teaching in this fashion is not as simple as direct learning from a textbook. This newer learning features thinking and discovery instead of memorization of facts previously established by somone else. Drill and practice are not abandoned, they come at the appropriate stage when insight has yielded a new conclusion.

Other teachers may not wish to use precisely the same procedures and devices that are used by the Boston group. Many similar materials can be developed. Most important, teachers must be a little imaginative and creative but this creativity should have a firm basis in understanding our number system and how numbers may be used in various relationships.



A Cross Number Puzzle for Flag Day

ACROSS

1. Our flag has ______ stars.
2. Flag Day is celebrated on June ______.
5. The beautiful tradition that Betsy Ross made a stars and stripes flag from a pencil sketch given to her by Washington in ______ has become a classic.
7. The flag has ______ red stripes.
8. The flag has ______ white stripes.
9. If Alaska and Hawaii become states, ______ new stars will be added to the flag.
10. The original stars and stripes flag has ______ stars.
12. The stars in our flag have ______ points.

DOWN

1. Seven less than the number of states in the United States.
2. 2001 divided by 23.
3. 4^2.
6. The Declaration of Independence was signed in 17______.
10. Since ______ an additional star has been added to the flag when a new state was admitted to the Union.
13. Independence Day is celebrated on July ______.

Contributed by MARGARET WILLERDING of San Diego State College, California

Arithmetic with Frames

UICSM PROJECT STAFF[1]
University of Illinois, Urbana, Illinois

THE PROJECT OF THE UNIVERSITY of Illinois Committee on School Mathematics is primarily concerned with students in grades nine through twelve. Frequently, the Project staff is asked if its work with high school students has implications for students in earlier grades, that is, if in attempting to work out better ways of presenting material to high school students, ideas have occurred for better ways to present mathematics to elementary school students. This article is an initial report of only a few such ideas. It should be pointed out that the authors have not tested these proposals in grade school classes as they have their ideas for high school students. However, we have done some informal experimenting with groups of two or three children ranging in age from 4 to 11 years.[2] We offer these suggestions not as a comprehensive new program nor as answers to any specific problems, but rather as informal notes for those teachers who are interested in trying new ideas. In the remainder of the article, for the sake of economy, we have omitted all tempering words; each declarative sentence is prefaced tacitly with some such expression as 'It seems to us that . . .' or 'Our guess is . . . '. Also we recognize that some teachers may respond to our recommendations with : "Why, I've been doing that for a long time." We send warm greetings to such teachers.

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At each stage of their learning of arithmetic, children should be taught to do some of the corresponding kinds of "algebra." In this way, algebra grows naturally out of arithmetic and would not be viewed in high school as an entirely new subject in which, too often, the student thinks he "adds and subtracts letters." If a student knows that

$$5+3=8,$$

then he is ready to be asked to replace the question mark by a *numeral*[3] in each of the following three incomplete expressions so that in each case the resulting assertion is true:

$$?+3=8$$

$$5+?=8$$

$$5+3=?$$

[1] University of Illinois Committee on School Mathematics—Project for the Improvement of School Mathematics: Max Beberman (Director), David A. Page (Editor). Herbert E. Vaughan (Mathematics Consultant), Gertrude Hendrix (Teacher Coordinator), *et al.* The ideas expressed in this article are to a large extent derived from the texts, teacher commentaries, and teaching program of the project. This article and the other work of the project has been made possible by a grant from the Carnegie Corporation of New York.

[2] Recently, the UICSM has begun experimenting with the instruction of gifted students who have completed the sixth grade and are enrolled in an accelerated junior high school program at the University of Illinois High School. These students are of seventh grade age and are working with mathematics materials which we have designed for the ninth grade.

[3] For explanations of our use of the word 'numeral' as distinguished from the word 'number' and of our use of semi-quotation marks, see *High School Mathematics—First Course* (Urbana, Illinois: UICSM, 1956) or the article "Words, 'Words' 'Words'." *The Mathematics Teacher*, April, 1955, p. 213.

Also, the student should be asked to make other replacements for each question mark so that the resulting statement is false. In an attempt to teach students correct arithmetic facts, many teachers make students feel that the writing of a statement such as:

$$5+6=8$$

is an immoral act. Rather than this, a student should examine such a statement and simply declare that it is false (or "not-true," as some children prefer to say). When the student gets to high school mathematics, he will no longer be able to live in a Utopia where false statements are never allowed to occur. Even in grade school work, he should be encouraged to make estimates and enlightened guesses and when he checks such a guess, he must be prepared to find that he obtains a false statement which tells him that his guess was not good enough.

★ ★ ★

An early elementary school student becomes too accustomed to the pattern:

Add:

$$\begin{array}{r} 2 \\ 3 \\ \hline 5 \end{array}$$

and should work with, at the same time, the pattern:

$$2+3=5$$

and should also work with the pattern:

$$5=2+3$$

One of the authors asked several second and third graders if the following statement is true:

$$2+2+2=2+8$$

A surprising fraction of them, "knowing that only one numeral can follow an equals-sign" transformed it mentally into '2+2+2+2=8' and said it was true. Students need to learn the symmetry of the use of '=' from the very outset of their work with it.

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Rather than using question marks to give incomplete expressions such as '3+?=5', we propose the use of large frames. Thus, students might be told to complete the following incomplete statements, first so that each of them becomes true, and then so that each of them becomes false:

Add:

$$\begin{array}{r} 3 \\ \square \\ \hline 10 \end{array} \qquad 4 + \triangle = 9$$

$$\diamond + 6 = 13 \qquad 10 = 2 + ⬡$$

After students have carried out these exercises, ask them to tell for a given exercise how many choices for replacement there are which will give true statements. If you want to introduce a bit of philosophical discussion in the arithmetic class, ask them to support the answer that there is only one number such that replacement of 'Δ' in '$4+\Delta=9$' by a name for this number makes a true statement. [A clever student may say that either '5' or '2+3' can go in the box to give a true statement. In this case he is giving two names for one number, just as 'William' and 'Bill' may be two names for one boy.] Also, ask them how many choices they have for replacements which give false statements. Accept answers such as "many" or better yet "as many as you please" or "so many you couldn't even write them all down" but do *not* introduce the word 'infinity.'

At a later stage, students can be introduced to expressions which contain more than one frame. For example:

$$\square + \square = 12$$

When more than one frame occurs, the student must learn an important rule of

the game. In the two boxes above, he must put names for the same number in each box. Thus, if he writes '3' in one of the boxes, to follow the rule he must write '3' (or '2+1,' etc.) in the other box; he must not write '3' in one of the boxes and '4' in the other box. (This rule of "like" replacements holds only for frames having the same shape.) Again, ask students to consider how many choices they have for making '□+□=12' into a true statement, and how many choices they have for making it false.

When students have had considerable experience in playing the "replacement game," they will find it convenient to use the word *satisfy* in such cases as:

the number 6 satisfies the expression

'□+□=12'

and:

7 does not satisfy '△+3=11.'

Notice that only one number satisfies '□+□=12' although any of that number's names could be used in satisfying the expression, *i.e.*, in changing it into a true statement. For example, if a student writes '5+1' in one box and writes '3+3' in the other box, he has followed the replacement rule correctly and has found that the number 6 satisfies the expression. At various stages of his work in arithmetic, it would be appropriate for a student to be confronted with problems such as the following.

For each expression, find all the numbers which satisfy it:

⬡ + ⬡ = 16

□ + □ = 15

1 = ◇ + ◇ + ◇

16 = □ × □

○ × ○ × ○ = 8

□ + □ = □

Such a list can be extended indefinitely and as much difficulty as is appropriate can be introduced using fractions, decimals, and "big numbers." Mixed in with such exercises should be some, such as:

5 + ◇ = 2 + ◇ +3,

which are satisfied by every number. The student has solved the problem when he points out that every number satisfies it (or that no number fails to satisfy it). The fact that every number satisfies such an expression can be disguised as intricately as you please by increasing the complexity of the expressions. For example:

16.2+□+8.3+□=□+36.9+□−12.4.

Also, among such problems should be some like this:

⬡ + 3 = ⬡

Here, the student should observe that there is no number which satisfies this expression and, correspondingly, every number will make it false.

Another interesting exercise is:

◇ × ◇ = ◇.

Here, the student should observe that the numbers 0 and 1 satisfy the expression and that no other numbers do. Encourage students to do a considerable amount of thinking in support of such a conclusion. Look for statements such as, "Well, if you multiply any number bigger than 1 by itself, you get an even bigger number. If you multiply a number between 0 and 1 by itself, you get a smaller number." Arrival at such conclusions can be promoted

through questions such as:

If you multiply 8 by a number larger than 1, what do you know about the answer?

If you multiply 1,632 by a number less than 1, what do you know about the answer?

In fact, as soon as a child can tell *without computation* which of the numbers $\frac{981}{972}$ and $\frac{981}{972} \times \frac{981}{972}$ is the greater, and whether $\frac{2355}{2356}$ is greater or less than $\frac{2355}{2356} \times \frac{2355}{2356}$, you know he has discovered the important ideas mentioned above whether he can give precise statements of them.

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If you give the student an exercise such as:

$$10 = \square + \hexagon$$

then, since you are using differently shaped frames, the rule is that he *may* use different numbers in making the replacements. Now candidates for replacement are *pairs* of numbers rather than single numbers. You might speak of candidates for the box as *first numbers* and candidates for the hexagon as *second numbers*, since in left-to-right order the box appears first and the hexagon appears second. Thus, the pair of numbers with first number 7 and second number 3 satisfies the given expression. Students will soon recognize that for every pair which satisfies the expression, the pair obtained by interchanging first and second numbers also satisfies the expression. This is because they believe:

Every replacement of

$$'\square' \text{ and } '\hexagon'$$

by numerals converts:

$$\square + \hexagon = \hexagon + \square$$

into a true statement of arithmetic.

This rule refers to the *commutative principle for addition*, one of several fundamental laws of our number system. [Students do not need to learn the word 'commutative' but they ought to discover or re-discover the universality of the rule.]

If you give students an exercise such as:

$$10 = \left[4 \times \square\right] + \hexagon$$

they will quickly find that it is not the case that for every pair that satisfies this expression, the "reverse" also does. [But you can ask them to find *a pair* such that it and its reverse both satisfy the expression. The pair (2, 2) is the only such pair.] Again, ask students to give pairs which satisfy the expression and pairs which do not satisfy the expression, and discuss the numbers of choices they have for pairs which do and for pairs which do not satisfy the expression. Problems of this type can be constructed with any desired degree of computational complexity.

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Once students are familiar with the ideas of working with these differently shaped frames and when they understand the replacement rule, you can use frames in many more places. For example, when introducing students to the ideas of fractions you can make precise the idea of, say, the fractional number $\frac{1}{3}$ by telling them that this is *the one number* which satisfies the expression:

$$\square + \square + \square = 1.$$

This is a step toward a mathematically sound formulation of the intuitive idea of

cutting a pie into three pieces of the same size. [We are not recommending that the pie idea be dropped but rather that it be used to support the statement above.]

The frame-notation provides a convenient tool for the introduction of *inverse operations*. Students can learn that the number

$$5-3$$

is the number which satisfies the expression:

$$5 = \text{⬡} + 3.$$

Thus, the student should realize that two ways of giving the same problem or, more precisely, two expressions which are satisfied by the same number are:

$$5 - 3 = \lozenge \quad \text{and:} \quad 5 = \lozenge + 3\,.$$

Similarly, when the student is introduced to division, he can learn that the number $6 \div 3$ is the number which satisfies the expression:

$$6 = 3 \times \square\,.$$

When you present the idea of square root, the student can learn that $\sqrt{9}$ is the number which satisfies the expression:

$$\text{⬡} \times \text{⬡} = 9.$$

You can move in another direction which is not common in grade school but which we think might be profitably undertaken, particularly in the later grades. You can give students problems which become more difficult as indicated by the following sequence. Of course, you would give many more problems than those listed, taking perhaps several weeks in going from problems like the first one to problems like the last one.

$$5 + \square = 13$$

$$10 = \text{⬡} + \text{⬡} + 3$$

$$\frac{1}{2} = \left[2 \times \lozenge\right] + \frac{1}{4}$$

$$\left[2 \times \square\right] + 6 = 12$$

$$\left[\frac{1}{3} \times \lozenge\right] - 4 = 1$$

$$\frac{\left[2 \times \square\right] + 3}{5} = 3$$

$$\left[2 \times \square\right] + 1 = \square + 4$$

You will recognize, of course, that students are solving equations in one variable.[4] Students enjoy working problems like these as straight puzzle problems:

"Can I find a number which satisfies it?".

Do not introduce general rules for solving equations. Let each student work out his own methods and discourage attempts to give procedures for obtaining answers, such as transposing or dividing both sides, etc. Tell students that a method which they work out for themselves will be more meaningful and longer remembered than one which they are told by a classmate. Again, it should be noticed that there is another method for giving practice in computation as well as preparing a student for a real understanding of algebra.

[4] We prefer the word 'pronumeral' to the word 'variable.' See *High School Mathematics—First Course, op. cit.*

The frame-notation[5] is suggestive of many game-type activities which should appeal to elementary school children. Here is one which includes a self-checking feature. The student is given a list of "equations" such as:

$$(1)\quad 4 + \triangle = 7$$

$$(2)\quad (9 \times \triangle) + (2 \times \square) = 37$$

$$(3)\quad 4 \times \square = (6 \times \square) - \diamond$$

$$(4)\quad (3 \times \diamond) + 12 = (4 \times \diamond) + 2\,.$$

His first task is to find a number which satisfies (1). He discovers that such a number is 3. So, he writes '3' in the 'Δ' of (2) and obtains:

$$(2')\quad (9 \times 3) + (2 \times \square) = 37$$

Next, he must discover a number which satisfies (2′); 5 satisfies (2′). So, he writes '5' in each '$\square$' in (3) and obtains:

$$(3')\quad 4 \times \boxed{5} = (6 \times \boxed{5}) - \diamond\,.$$

10 satisfies (3′). Expression (4) is so constructed that it is satisfied by 10.

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As indicated in our opening remarks, the suggestions made in this article do not prescribe a new mathematics curriculum for the elementary school. But they do reflect the belief that children are willing to spend time on mathematics for the sheer intellectual challenge in can be made to offer. Such challenge is present whenever a child is encouraged to use his imagination and intuition. Games which are rich in mathematical (and not necessarily complex computational) content always contain challenges to which children respond. Perhaps we teachers are spending too much time fretting over what shall be done for the gifted few instead of developing stimulating techniques to amplify the mathematical content of the curriculum for all children.

[5] In preparing lists of exercises in duplicated form (by spirit or stencil duplicating processes) you will find it convenient to use a template containing various sizes of frames (circular, triangular, square, and hexagonal). You can make such a template out of stiff cardboard or obtain one from a drafting supply house. ["RapiDesign Pocket Pal No. 50" is an excellent template; it is manufactured by Rapi-Design, Box 592, Glendale, California. Another is the "Draftmans Do-All Template No. 433D" Alvin and Company, Windsor, Connecticut.]

EDITOR'S NOTE. While it may appear that the authors of the "Illinois Project" are assuming that the mathematics of the elementary school is primarily a preparation for the mathematics of the secondary school, such is not the case. They are genuinely interested in a better elementary school arithmetic. Through the introduction of "frames" they offer a convenient and intriguing device for stimulating thinking and discovery. They very carefully avoid formalism and the typical equation form and at the same time have introduced some excellent equation thinking. They are first to admit that many good teachers have been doing similar things without the convenience of the frames. We believe that the larger the element of thinking and discovery in learning, the longer this learning will remain with the learner and perhaps also the method will "transfer" or be more readily extended and enlarged than learning which is largely rote memorization.

We are all watching the experimental work being done at the University of Illinois High School. This project may well have a considerable influence not only upon secondary school education but also upon elementary school and college. Let us as teachers weigh carefully new ideas and not be afraid to try them with our own pupils.

Summer Meeting

August 18–21 at Northfield, Minn.

In July all members of the National Council of Teachers of Mathematics will receive copies of the program for the summer meeting to be held on the Carleton College campus. A dozen discussions on various topics in arithmetic will be featured. Blanks for room reservations will be enclosed with the programs. It is always a pleasure to meet on a college campus where both ideas and money seem to go farther than in a commercial establishment. We will be looking for you in MINNESOTA IN AUGUST.

An Approach to Problem Solving

EDNA WISELY
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PROBLEMS—story problems—experience problems—sentence problems—word problems are all used to designate a situation in which children must read and then decide which process or processes are to be utilized in arriving at the solution of the question proposed. To the adult this seems a relatively uncomplicated matter especially if he does not recall clearly his own school days. Unfortunately many children approach problem solving in a state of bewilderment induced by fear and failure. Here are no neat little sign or directive words to offer security to the agile manipulator of number. The child is truly confronted by "problems" not the single problem which is stated. His problems often are: Do I add? Subtract? Multiply? Divide?

In discussing with a fourth grade their likes and dislikes in arithmetic, there was complete agreement that they disliked "those pages of sentence problems." Further questioning as to why they felt the way they did brought forth the following comments:

I don't understand them
I never know what to do
They don't tell us if we should add or subtract
I don't know if I should use all the numbers which are in the sentences
Sometimes there are words I don't know
Who thought-up those problems?

Children for many generations have shared the feelings expressed by these boys and girls. Many a large, red, soul-satisfying hundred which we received on our arithmetic papers was directly attributable to the answers in the back of the book. Our technique for arriving at the correct answer left much to be desired. We hurriedly read the problem. Then found the answer. If the answer was smaller than one of the given numbers, we divided. If it were considerably larger, we multiplied. Then if the process selected did not result in the given answer, we blithely selected another process. So through sheer elimination we eventually arrived. Little thinking entered into such lessons. Today's children have no recourse to answers in the back of the book!

Problems Are Important

What can be done so children are able to approach problem solving intelligently and confidently? *Unless children learn to solve problems the purpose of teaching arithmetic has not been achieved.* Skill in computation is separated from problems only on the pages of our arithmetic books. As important as the ability to add, subtract, multiply and divide is, without understanding of when to apply these specialized skills they have little value.

The United States Army was well aware of the inability of enlisted men to cope with quantitative situations. Men evidenced a degree of skill in computation, but when the same number were placed in a descriptive situation which a soldier might encounter, they were in doubt as to the process or processes to use in the solution. The Army concluded that computation had been stressed, but that accurate, skillful thinking in mathematical situations had been neglected. The tricks with numbers which had been learned proved to be useless under conditions of use.

Many adults are keenly aware of their inability to function efficiently with number situations. Schools need to consider new techniques for improving problem-solving. Who has a better laboratory in which to try new ideas than the class room teacher?

A group of first-grade children engaging in wood construction during the social studies period had no difficulty in recognizing the function of the tools the first time the tool chest was wheeled into the room. A saw was to make a board shorter. A plane was to make a board smoother. A hammer was to drive nails or pull out nails. Each tool was identified in terms of its special task. No child attempted to use a tool to perform an inappropriate bit of work. Lack of skill in using the tools was evident, but the function of each tool was clearly understood.

Tools Are Valuable if Used

Children learn to manipulate our four mathematical tools—addition, subtraction, multiplication, and division—without ever knowing their specific functions. Is it possible for children to use these mathematical tools with the same degree of understanding that the first graders used carpenter tools? With this problem in mind, the writer arranged to work at several grade levels to determine children's degree of understanding of the functions of the processes and how they could be helped to gain greater insight.

In working with a group of fourth grade children this line of reasoning was developed by discussing tools used by carpenters, housewives, farmers, and doctors. The children were interested in considering that there are tools which belong to each worker. The group decided that a good workman in using his tools must know:

A GOOD WORKMAN

First: What he wants to accomplish
Second: What tool is needed
Third: How to use the tool

These three statements were listed on the board. When children were asked to name the four mathematical tools available for use in solving problems, they named such articles as pencil, eraser, ruler or paper. After the suggestion was made that these couldn't be considered mathematical tools as they were used in other subjects, the children named addition, subtraction, multiplication, and division. These four words were written on the board. The class agreed if these four words were names of tools each must have a function or functions. A discussion to determine the understanding of the function of addition followed. When answers such as the following were given by the children the lack of understanding was evident: to find the sum; to carry from one's place to ten's place; to add; to find the answer; to add; and, to write numbers in a straight row. Bear in mind that these children had been using *addition* for at least three years, and no one could offer an answer which showed a glimmer of understanding. The next step was to help children think through experiences so they could formulate a statement of their own regarding the meaning of addition. To do this children needed to be free of fear of making mistakes, and willing to put their thinking into words.

Children were placed in groups in the front of the room. The size of each group and the number of groups were noted. Then one large group was formed. This problem was worked at the board. Following this many oral problems were presented to the children. These problems involved social situations which were in the experience of the children. Each time attention was called to the number of groups, the size of each group, and the comparative size of the total group. Children were encouraged to make up oral problems and question the class as to the number of groups, the size of each group and the size of the combined groups. When the question as to the function of addition was again asked the children concluded that *Addition is a tool used to put or "think" smaller groups into one large group.*

From the discussion it was evident that children now understood that addition is a grouping process that does not result in "more."

The development continued on the following day when the question of the purpose or purposes for which the tool multiplication is used was asked. Again it became evident that although the children were skillful in computation with multiplication they were unaware of the function of the skill. A simple multiplication problem was presented orally and worked on the board by both addition and multiplication. Attention was focused on the size of the groups. Many oral problems were presented by the teacher and children. Finally the children were able to generalize that *Multiplication is a tool used to put or "think" smaller equal groups into one large group.* In comparing the statements regarding the addition and multiplication, the children discovered that only one word *equal* was different.

The children decided that if addition and multiplication are used to regroup small groups into one larger group then subtraction and division must be used to separate a large group into smaller groups. After many oral problems, considerable discussion, and much directed thinking the class reached these conclusions:

Subtraction is a tool used to separate a large group into two small groups. One of the smaller groups is known.

Division is a tool used to separate a large group into two or more smaller equal groups.

The question as to what can be done besides combining groups and separating groups was asked. The children were nonplussed by this until a simple problem was offered:

John is in the fourth grade. His best friend, Henry, is in the same grade. When the nurse weighed the children, John's weight was 87 pounds. Henry weighed 79 pounds. How many pounds more than Henry did John weigh?

The children read the first statement on the chart "A Good Workman" and agreed that they must decide what they wished to do before they selected a tool to use. The children quickly decided that the two weights were being compared and that subtraction was the proper tool to use. As these children were just beginning a study of division, comparison of two groups by division was not developed until later.

The children's discoveries were charted in this form:

Work of Arithmetic Tools

Putting Groups Together

Addition is used to form a large group from two or more smaller groups.

Multiplication is used to form a large group when all the smaller groups are equal in number.

Taking Groups Apart

Subtraction is used to separate a group into two smaller groups: One of the smaller groups is known.

Division is used to separate a group into two or more smaller equal groups.

Comparing Groups

Subtraction is used to compare two known groups.

This chart was used for reference purposes with the class.

The function of the additive process was readily grasped by a group of first grade children. Six children were formed into three small groups. Attention was directed to the size of the groups and the number of groups. The number story was read from left to right: 1 child and 3 children and 2 children are 6 children. When the magic word "are" was said the children quickly formed one group. All possible groupings were made with the 6 children. The 6 children enjoyed planning the story, grouping themselves, and selecting a friend to read the story. Everyone in the class participated as one of the six. Attention and interest did not lag. Sometimes the stories were stated on the board in picture form.

♀ ♀ and ♀ and ♀ ♀ ♀ are 6

More mature children used the number symbols:

2 and 1 and 3 are 6

At the close of the lesson the teacher said, "When you go home, you may want to tell your mother what we have been doing. This kind of arithmetic is called *addition*. If your mother should ask what addition is, what will you tell her?" The first child who answered said, "I'll say that addition is used to put small groups together to make one large group." Could there have been a better answer? The child defined addition in terms of its function. If children were helped to recognize the function peculiar to each of the processes at the time of their development, would they be more confident in problem solving?

Problem Development Continuous

No teacher will interpret the above question to mean that the functions of the processes are developed during the initial presentation and then forever afterwards remembered. There must be a continuing development and recall as each new step of a process is brought into focus. Addition is a tool used to put or "think" groups of common fractions or decimal fractions together as well as whole numbers.

The place of problems in developing the ideas charted can not be over stressed. In using materials whether that material be children, chairs, pictures on a flannel board, birthday candles, or dolls, problems need to be woven about them. The teacher might offer several story problems and then give children an opportunity to use their imaginations to make up problems. Many teachers keep a basket of small objects such as paper dolls, jacks, pennies, racing cars, plastic chairs, and balls to stimulate problem making. Forming groups, separating groups, and comparing groups becomes understandable when objects are manipulated.

Problems without number appeal to children and offer a basis for consideration of what should be done without the complication of number.

EXAMPLE:

Each class in the school was collecting money for the Red Cross. Children brought any amount they wished. At the end of the week each teacher counted the money collected in her room and sent the money and a slip stating the amount to the principal. When all the classes had sent in their money, the principal placed a notice on the bulletin board telling how much the school had collected. What tool did the principal use to find the amount collected?

The children realized that in such a problem groups were being put together so either addition or multiplication would be used. One boy remembered that the problem stated that each child brought any amount he wished so he argued that all the rooms would not be apt to collect the same amount so the small groups would be put together by addition.

If children can learn to consider problems in terms of the three possible actions in regard to groups they are well on the way to the solution. Having decided what is to be done with the groups, then they must consider the nature of the group. This should result in the selection of the correct process, and, if the process is used SKILLFULLY, the correct answer. *If success is repeated often enough, confidence in one's own ability should replace the attitude of fear and confusion too often evoked by problem solving.*

EDITOR'S NOTE. The editor is a little disturbed by reference to the skills and computational processes of arithmetic as *tools*. He regards arithmetic computation as much more than a mere tool that is available for use when the occasion demands. There is a good deal of association value in the stages of learning to *do* and *use* arithmetic and that is what Miss Wisely is striving for in her approach to problem solving. If a skill-tool is all that teachers wish in their pupils, we might better buy a little computing machine and give to each youngster and then teach him how to use it. The net cost would probably be less than our present method. It is the elements of *thinking* and *understanding* that we strive for and it is these that distinguish the human from the machine.

Shall We Expose Our Pupils to Errors?

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WILL EXPOSING PUPILS TO ERRORS generally made in the addition and subtraction of fractions result in a lower level of achievement than if conventional procedures are followed? Certain studies in typewriting, spelling, language, punctuation, and speech have claimed that an exposure to errors will either result in adverse effect on the achievement level or will result in a higher level of achievement. As no rigorous studies related to this problem in the area of arithmetic have been reported in the literature, the author in a doctoral dissertation[1] determined to what extent the identification and correction of typical errors embedded in practice exercises in addition and subtraction of fractions affected learning of these processes.

Each set of practice exercises was systematically constructed in such a manner that every exercise contained examples in the steps that had been taught in all previous lessons in the particular process. The errors introduced in the experimental exercises were based on previous studies of the frequency and persistency of errors in the addition and subtraction of fractions by the author.

Pupils in experimental groups were required to identify and correct typical errors embedded in worked-out examples, while the control groups worked the same examples as practice exercises. Nineteen practice exercises of twenty examples in each were prepared for the experimental groups as well as the control groups. The example below illustrates a typical error that was to be detected in Experimental Practice Exercise No. 16—Subtraction:

$$\begin{array}{r} 8 \\ 1\frac{1}{2} \\ \hline 7\frac{1}{2} \end{array}$$

Seven classes with 289 sixth grade pupils from the Robbinsdale, Minnesota, Public Schools participated in the main study that began Nov. 10, 1952, and ended February 19, 1953. Pupils in two classes were given systematic practice in identifying and correcting errors embedded in selected examples, *Method A*. Pupils in two other classes served as controls. They merely worked the same examples as practice exercises, *Method B*. In order to control the teacher factor, pupils in three classes were each divided randomly into two sections. One split-group was subjected to Method A, while the other was subjected to Method B.

This?

Detect and correct the error:

$$\begin{array}{r} 3\frac{9}{16} \\ +6\frac{3}{16} \\ \hline 9\frac{12}{16} = 9\frac{4}{8} = 9\frac{1}{2} \end{array} \quad 9\tfrac{3}{4}$$

Or This?

Work the example:

$$\begin{array}{r} 6\frac{3}{4} \\ +2\frac{1}{4} \\ \hline 9 \end{array}$$

The tests used to evaluate outcomes included: (1) The Kuhlman-Finch Intelli-

[1] "The Effect of the Systematic Analysis of Errors on Achievement in the Study of Fractions at the Sixth Grade Level," Unpublished thesis, Minneapolis: University of Minnesota Library, University of Minnesota, 1953, 324 pp.

gence Test—Grade VI; (2) The Coordinated Scales of Attainment—Arithmetic Computation; and (3) Brueckner's Comprehensive Tests in Addition and Subtraction of Fractions. The validity and reliability of the measuring instruments met the criteria established by the author.

The study completed under the guidance of Dr. Leo J. Brueckner and Dr. William Moonan was designed to test the following null hypotheses:

1. There is no difference between the all-experimental[2] and the all-control[3] groups in their ability *to add* fractions after the experimental group has systematically analyzed errors in the same process while the control group has not made such analyses.
2. There is no difference between the all-experimental and the all-control groups in their ability *to subtract* fractions after the experimental group has systematically analyzed errors in the same process while the control group has not made such analyses.
3. There is no difference between the split-experimental[4] and the split-control[5] groups in their ability *to add* fractions after the split-experimental group has systematically analyzed errors in the same process while the split-control group has not made such analyses.
4. There is no difference between the split-experimental and the split-control groups in their ability *to subtract* fractions after the split-experimental group has systematically analyzed errors in the same process while the split-control group has not made such analyses.

[2] Classes in which all pupils received the control treatment.

[3] Classes in which all pupils received the experimental treatment.

[4] The section of a class (randomly divided into two sections) in which the pupils received the experimental treatment.

[5] The section of a class (randomly divided into two sections) in which the pupils received the control treatment.

Each of the above hypotheses was tested in two ways: 1) in terms of the criterion variable based on an immediate recall test, and 2) in terms of the criterion variable based on a delayed recall test. A survey of the research available related to the Beta-hypotheses identified by Knight Dunlap and to various error studies in arithmetic revealed that in general the studies lacked adequate controls, randomization, and replication.

When the analysis of variance and covariance was applied to the data, the following conclusions were drawn from the findings:

1. No statistically significant differences due to method (i.e., to the two types of treatment) were found in the experiment in the addition of fractions either in immediate or delayed recall.
2. In the experiment in the subtraction of fractions statistically significant differences due to method were found in the delayed recall tests when the teacher factor was *NOT* controlled. When the teacher factor *WAS* controlled, statistically significant differences due to method were found in in the immediate recall tests in the subtraction of fractions. Thus after a prolonged exposure to error analysis adverse effects were noted in the later stages of learning subtraction of fractions.
3. When the pre-test grand means for the control groups and experimental groups were compared with the post-test grand means, marked growth of achievement in both processes was evidenced.

Implications for Teaching

The outcomes of this experimental study in learning suggest that in classroom procedures the following new avenues of approach are possible in the teaching of addition and subtraction of fractions:

1. The exposure of children to the process of identifying and correcting errors in practice exercises in addition and subtraction of fractions will not affect learning adversely in the early stages of learning in these processes.
2. In order to provide a wide variety of learning activities in the classroom, teachers may include experiences in detecting and correcting errors systematically in these processes on a group or on an individual basis. This procedure may have no effect during the later stages of learning an operation.
3. Instruction in the addition and subtraction of fractions based upon the procedures outlined in the basic text used, *Arithmetic We Use—Grade 6*, and upon systematic practice—exercises prepared for this experiment resulted in post-test mean scores for both the experimental and the control groups that were actually within a few raw score points of complete mastery.

Is it possible to expose our pupils to errors in addition and subtraction of fractions and not effect learning adversely? The answer is—YES.

EDITOR'S NOTE. Dr. Aftreth found that exposure to error produced no adverse effects upon final learning in his experiment involving addition and subtraction of fractions. He used the procedure primarily as a learning device or more properly as a stage in learning. He has given no conclusions relative to the facility with which either group detects errors. It may be pointed out that one of the real functions of arithmetic in our society is to enable people to follow the reasoning and the computation of another and to detect error or flaw in any stage of the argument or computation. Perhaps this is a function that can be taught by means other than the usual method of learning to do things correctly. We need additional research on this point.

Many teachers have used "error" as an "interest-impetus" device with their pupils. There seems to be a great delight when a pupil finds that a teacher has made an error (deliberate we hope). It would seem also that detection of error ought to be based upon sound basic learning. Many teachers still feel that it is wrong to expose young children to error because they have an idea that the initial response should be a correct response especially if it is one that is expected to be learned and remembered. What do our current psychologists say about "the potency of the initial response"? Is it stronger and more lasting than subsequent responses? We have numerous examples of some one making a mistake which he later corrected and when he later made a mistake in the same situation he returned to the original error and not to some other.

Council Announcements

NEW OFFICERS

The partial slate of new officers which was elected by mail last month includes:

Vice Presidents

Robert E. Pingry (College)
University of Illinois
Alice M. Hach (Junior High)
Racine, Wisconsin

Board of Directors (3 year term)

Clifford Bell
University of California at Los Angeles
Robert E. K. Rourke
Kent School, Kent, Conn.
Annie John Williams
Durham, North Carolina

More Nominations Needed

The Committee on Nominations and Elections is anxious to have suggestions from Council members for candidates for the 1958 election. New officers to be elected include: president, vice-president for high school, vice-president for elementary school, and three directors. Suggestions may be sent to the chairman of the committee: Dr. Clifford Bell, Department of Mathematics, University of California, Los Angeles, California. Send suggestions before May 28, 1957.

Mental Arithmetic

Donald W. Lentz
Prin., Port Washington Jr. H. S., New York

At a recent social gathering one of the guests was describing an experience involving an eminent scientist. "It was unbelievable that he could think through difficult higher mathematical computations without writing a thing, become so absorbed in the process that nothing else mattered, and after a period of mental gymnastics 'out of this world' he would set down a tentative result with a look of triumph and success in his eyes. It was wonderful to see."

The "mental mathematics" of this scientist are a far cry from the simple mental arithmetic of a little child who learns to think that $2\times2=4$. But between these two extremes lie countless opportunities for the ordinary application of mental arithmetic. And the pity of it is that so little time is devoted to practice with mental arithmetic, and there is oftentimes so little realization of the numerous areas for its application. *Without a doubt, for the most of us mental arithmetic or mental mathematics can be a most efficient mode of computation within our respective limitations.*

Where is it used? Mental estimation and approximation can be done rapidly. The cost of food purchases may be quickly computed accurately or approximately. Intermediate steps in multiple step computations can be worked faster. The selection of a correct arithmetic or mathematical process requires a brand of mental arithmetic.

Trial partial quotients? Trial common denominators? These operations can be speeded by eliminating a "jotting down" step. With a moment's thought anyone can list many uses for mental arithmetic from "How long will it take me to drive to Philadelphia?" to "About how much will my income tax be?"

What can we do about it? Give the children more opportunity to do arithmetic problems without using pencil and paper. Practice! Practice! Practice! Use abstract numbers or concrete situations. Drills with abstractions can be fun and can be profitable. Practice with concrete situations can improve skill and build computational confidence in "real" cases. For example, Miss A uses a quick drill at the end of each class period, firing problems in fundamental processes as quickly as the class can follow. Mr. X uses part of each week for "mental estimations" with concrete cases. Miss B makes sure that every class solves some "story" problems periodically without use of pencil and paper. The pupils like the change of pace!

This is not a question of "training the mind" to be mathematical. It is more a matter of establishing self-confidence and self-reliance and to develop a realization that everything need not be written in computation. Depending upon the ability of a given pupil or class, once basic understandings and concepts are established pupils may be encouraged to eliminate the writing of steps and to do the computations in their heads.

Who can learn mental arithmetic? Everybody can, within the limitations of native ability, granting that adequate opportunities are provided for practice. Thus it would be foolhardy, indeed, to expect an inept pupil to attain the mental goal reached by the scientist who impressed our guest; but, on the other hand, it would be equally absurd to assume that a student of reasonable potential couldn't develop in mental arithmetic ability and thereby increase his computational efficiency.

Where do we begin? Practice with mental arithmetic can begin as early as children begin to develop quantitative concepts and skill can be improved throughout the entire mathematics career.

Mental computations have a definite place in our quantitative lives and it is our responsibility to provide practice and opportunity to enable every pupil to increase his computational efficiency through this simple technique.

Report of the Membership Committee

Mary C. Rogers, Chairman
Roosevelt Junior High School, Westfield, N. J.

The membership committee of the National Council of Teachers of Mathematics is very happy to report to you that the 15,000 membership goal toward which we have been working during the past two years has been reached. The membership count of December 17, 1956 received from the Washington Office indicates a total of 15,205 members. This achievement was received with considerable enthusiasm by the National Council Board and by many other persons in attendance at the Jonesboro meeting. We are confident that you, too, will be keenly interested in this news. It has been the assistance of you as individuals which has largely made this accomplishment possible. The Membership Committee and the National Council Board are indeed grateful to you for your fine support and helpfulness.

Record of Membership Growth

We have prepared the accompanying membership analysis based on the December 17, 1956 count received from the Washington Office. You will be interested in the part your State and you have played in bringing about this excellent membership growth.

Plans for the Future

We believe this record of achievement is one of which each of us should be justifiably proud. In the majority of cases, however, your local leaders have stated to the membership chairman that there are still a great many mathematics teachers throughout the country who are not being reached by the services of the National Council. These leaders are very desirous of accomplishing two things:

Leaders in Membership Totals

1. New York	1,275	6. Ohio	710
2. Illinois	1,094	7. Michigan	633
3. California	1,002	8. New Jersey	570
4. Pennsylvania	947	9. Indiana	532
5. Texas	755	10. Wisconsin	491

Leaders in Membership Growth

(Since December 1955)

1. Pennsylvania	166	6. Illinois	133
2. New York	157	7. Foreign	131
3. California	149	8. New Jersey	129
4. Texas	137	9. Kansas	112
5. Wisconsin	135	10. Arizona	90

Greatest Relative Growth

(Since December 1955)

1. Arizona	6. Canada
2. Idaho	7. Wisconsin
3. Foreign	8. New Hampshire
4. Kansas	9. New Jersey
5. Colorado	10. Mississippi

Analysis of Membership Growth—December 1955 to December 1956

	Dec. 1955	Dec. 1956	Goals	Per Cents
1. Alabama	130	144	170	85%
2. Arizona	68	158	60	263%
3. Arkansas	98	125	171	73%
4. California	853	1,002	879	114%
5. Colorado	155	222	180	123%
6. Connecticut	215	220	228	96%
7. Delaware	60	66	71	93%
8. District of Columbia	189	187	185	101%
9. Florida	322	359	351	102%
10. Georgia	160	164	186	88%
11. Idaho	13	21	18	117%
12. Illinois	961	1,094	1,188	92%
13. Indiana	488	532	542	98%
14. Iowa	272	292	296	99%
15. Kansas	207	319	311	103%
16. Kentucky	91	109	123	89%
17. Louisiana	220	260	276	94%
18. Maine	66	64	69	93%
19. Maryland	341	274	285	96%
20. Massachusetts	392	421	440	96%
21. Michigan	572	633	600	106%
22. Minnesota	301	375	391	96%
23. Mississippi	104	133	123	108%
24. Missouri	258	310	330	94%
25. Montana	58	62	60	103%
26. Nebraska	121	136	165	82%
27. Nevada	20	21	14	150%
28. New Hampshire	61	81	65	125%
29. New Jersey	441	570	561	102%
30. New Mexico	83	87	89	98%
31. New York	1,118	1,275	1,181	108%
32. North Carolina	207	196	255	77%
33. North Dakota	31	35	44	80%
34. Ohio	624	710	740	96%
35. Oklahoma	208	226	237	95%
36. Oregon	270	209	134	156%
37. Pennsylvania	781	947	920	103%
38. Rhode Island	50	58	71	82%
39. South Carolina	96	105	137	77%
40. South Dakota	51	55	36	153%
41. Tennessee	181	202	246	82%
42. Texas	618	755	672	111%
43. Utah	56	61	48	127%
44. Vermont	33	37	41	90%
45. Virginia	265	316	357	89%
46. Washington	233	265	228	112%
47. West Virginia	77	91	188	48%
48. Wisconsin	356	491	420	117%
49. Wyoming	36	30	36	83%
TOTALS	12,611	14,505	14,388	101%
U. S. Possessions	92	90	74	122%
Canada	182	258	201	128%
Foreign	221	352	339	104%
GRAND TOTALS	13,106	15,205	15,002	101%

1. Maintaining the generous membership which now exists through a high percentage of *renewals*.
2. Continuing and extending membership publicity with an expanded stimulation of interest in National Council serves which should result in many more new members for the Council.

We suggest that future procedures should continue to follow closely those you have found effective in the past.

1. We have found the "Each One Win One" technique to be our most valuable aid. We urge a continuance of its use by *all present members of the Council.*
2. Your *prompt renewals* greatly facilitate the keeping of records in the Washington Office and the preparation of reports.
3. You who are members of the mathematics staff in colleges of education and similar education centers can be —and are—of invaluable assistance through *your continued stimulation of interest* in NCTM services among your students.
4. Similarly, *supervisors of mathematics and/or mathematics department chairmen* are obtaining increasingly fine results in your work with your teachers. Keep up the good work.
5. *Many state and other local associations* of mathematics teachers, together with the *state representatives* to the National Council are performing an outstanding service to the Council. A continuance and expansion of your fine work will be most sincerely appreciated.
6. Library and other institutional subscriptions will still be included in preparing reports of membership totals.

The personnel of the Membership Committee continues much the same as during the past two years. There have been a few changes, however. The persons presently serving in this capacity are:

Janet Height
: New England States

Faith Novinger
: Delaware, District of Columbia, New Jersey, New York, Pennsylvania

Bess Patton
: Florida, Georgia, Maryland, North Carolina, South Carolina, Virginia

Pearl Bond
: Alabama, Louisiana, Mississippi, Texas

Marian C. Cliffe
: Arizona, California, New Mexico, Utah

Harold J. Hunt
: Idaho, Montana, North Dakota, Oregon, Washington

Lucille Houston
: Michigan, Minnesota, Ontario, South Dakota, Wisconsin

Nellie Alexander
: Illinois, Indiana, Iowa, Ohio, West Virginia

Florence Ingham
: Colorado, Kansas, Nebraska, Oklahoma, Wyoming

Nellie Kitchens
: Arkansas, Kentucky, Missouri, Nevada, Tennessee

These persons are all doing a most outstanding service for National Council in their work with State Representatives and with Presidents of Affiliated Groups in their respective territories.

We shall keep you informed of membership progress as often as information is available to us. We welcome your advice and suggestions at any time for the improvement of the work of the Membership Committee.

Please accept our best wishes for your greatest professional success at all times.

From Cake to Cancellation

Brenda C. Lansdown
Brooklyn College, Brooklyn, N. Y.

Concrete materials can be an aid to mathematical discoveries for children of all levels of ability. Materials have often been referred to as "crutches" to be relegated to the mentally lame. If we use instead, the analogy of the toddler's "walker," to represent a stage of intellectual growth, and leave the crutch for "remedial" work, then both take on a new meaning: throwing away the aid before the individual is ready may leave him permanently crippled. And who but a disturbed person hangs onto crutches and walkers when these are no longer needed?

Children between the ages of eight and eleven thrive, mentally and emotionally, on "fraction pies." These may be simply, bright colored construction paper circles of four inch radius. The child is given the circles and he cuts his own pies: folding across the diameter for halves, folding these again for *fourths* (not *quarters* at first, for quarters mean *twenty five* cents or *fifteen* minutes), and again for eighths. One must start the thirds' group for the children with a template, and from there they make the sixths. The very making of these fraction pies tells the children how many parts there are in a whole each time. One can start any lesson by asking for this information.

Experimentation begins as soon as a child has amassed two halves and four fourths. "In what ways can you make a whole?" starts the class off on its adventures. And because each wants to contribute, even after the easier possibilities have been exhausted, children tell how they make two wholes, or one and a half; they may cover up part of a whole and discover subtraction.

When the drive to discover is at an end, the results may be recorded on the chalkboard:

$$\tfrac{1}{2}+\tfrac{1}{2}=1 \quad \tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{2}=1 \quad 2-\tfrac{1}{4}-\tfrac{1}{4}=1\tfrac{1}{2}$$

At another lesson the eighth is introduced and the same procedure followed. However, there are now many more discoveries of combinations and the interest may last during several lessons. Cooking problems with measuring cups and water or sand may be worked out, doubling recipes, sharing a cup of lemonade between two, three, or four children. At this point it is interesting to note that some children will do the sharing physically with cup and water, some will use their representative material, (the fraction pies) and some will think in their heads or write the algorism as an aid. Each one uses the *right way for himself*.

At every point the teacher "comes from Missouri," except that the children are not showing her, but each other. "Tell us how you got it" "Can anyone explain it another way?" These questions are asked over and over until everyone is on his toes figuring. (It used to be called "drill," but the *thought* had to be removed first!)

In remarkably few sessions, average nine and ten year olds are ready for the "thirds-sixths" series. The same method is used. Soon the children will mix their two batches of fraction pies and find that two fourths are the same size as three sixths, but they will be frustrated about the limitations of combining thirds and fourths. Usually someone says, "The sixth won't fit in. About half of it will." The teacher asks, "What do you think half of a sixth will be?" It is a rare occasion when the child cannot reply, "Must be a twelfth."

Homemade twelfths are not very satisfactory because slight inaccuracies in cutting lead to faulty conclusions. Even sixths and eighths sometimes squeeze into each other's beds, the unaligned radii escaping notice. At this point the children can be motivated to do more of their thinking in algorisms or they can be introduced to the stimulating commercial Fraction Pie Game.[1] A spinner indicates to each player what sized rubber fraction he may pick up. He tries to fill his "wholes." The children play the game endlessly, far, far longer than any drill could be endured. They search for pieces to fill their gaps, large ones for quick winning; they substitute four twelfths for one third "automatically" (the way an astronomer may know the log tables!); they compare, "If I had a fourth and a twelfth (looking at the gaps in two "wholes") I'd be up to Kim. Gee! That's a third. Maybe I can exchange." Blessed word that, *exchange*.

Exchange is the concept behind cancellation. Children learn to exchange four sixths for two thirds, nine twelfths for three fourths, first of all with the materials, they directly from the experience and finally from the algorism. One can write such problems as these on the chalkboard:

$$\frac{4}{8}=\frac{1}{2} \qquad \frac{3}{6}=\frac{1}{2} \qquad \frac{6}{12}=\frac{1}{2} \text{ etc.}$$

and ask the children whether they can figure out a way to do these without using materials. *Some* but not *all* will be able to articulate that, "The numerator is half the denominator." Those not sure continue using materials and drawing on experience while they grow to the realization that there is a quicker way.

With:

$$\frac{4}{6}=\frac{2}{3} \qquad \frac{9}{12}=\frac{3}{4}$$

one has to sharpen the hypothesis. "Well, you can divide two into both." But this is a later product, slowly achieved. At first the child thinks aloud in this manner: "You've used pies which are twice as big" (meaning the thirds are twice the size of sixths), so you don't need so many of them. You need half as many." In the second example: "It takes three twelfths to make one fourth, so nine twelfths . . . well, that's three times as much, so it's *three* fourths." When the teacher asks, "Has anyone another way of doing it?" so many ingenious ways of looking at the problem are given that she has to keep on her toes to follow these little "Einsteins."

The eight-to-eleven-year olds, (from Miss Grace Cohen's class of the Downtown Community School of New York City,) who demonstrated their thinking for Section 10 of the N.Y.S.E.S.E. on Dec. 7th 1956, after only nine sessions of fraction teaching, plus free play with the Fraction Pie Game, produced this type of work.

I asked the children to make wholes with their pies any interesting way they could. A typical example was, "I made my whole with two twelfths, two sixths, one fourth, and two eighths." The girl wrote on the board:

$$\frac{2}{12}+\frac{2}{6}+\frac{1}{4}+\frac{2}{8}=1$$

I asked her to explain why she thought she was right. Her reply went like this: "I can exchange for twelfths. Here's two twelfths, and a sixth is two twelfths, so two sixths is four twelfths. That (the two twelfths and two sixths) makes six twelfths. Now one fourth can be exchanged for three twelfths and that's nine twelfths altogether. Two eighths. H'm. I can't exchange eighths for twelfths, but two eighths is a fourth and a fourth is three twelfths. Nine and three make the twelve twelfths. A whole." All done in front of a large audience of learned mathematicians and a tape recorder. When children can figure like this in nine easy lessons, there must be something to the method!

[1] Courtesy of *Creative Playthings*, 5 University Place. NYC 3.

Mathematics Institutes

EIGHTH ANNUAL MATHEMATICS INSTITUTE OF LOUISIANA STATE UNIVERSITY. June 23–29, 1957. Louisiana State University, Baton Rouge, Louisiana.

For further information write to: Houston T. Karnes, Louisiana State University, Baton Rouge, Louisiana.

WORKSHOP IN MATHEMATICS EDUCATION FOR ELEMENTARY SCHOOL, JUNIOR HIGH AND SENIOR HIGH SCHOOL TEACHERS AND ADMINISTRATORS. (Three semester hours of credit.) August 5–24, 1957. Northwestern University. Evanston, Illinois. Director: E. H. C. Hildebrandt.

For further information write: Dean E. T. McSwain, School of Education, Northwestern University, Evanston, Illinois.

In addition to the above, many colleges are offering mathematics workshops under the sponsorship of The National Science Foundation. The directors of these workshops usually mail announcements to teachers in their area. The institutes and workshops are good places to learn the newer developments from leaders in the field and to exchange ideas with other teachers.

MINNESOTA WORKSHOP IN ARITHMETIC

The University of Minnesota will hold a special workshop in arithmetic for elementary school supervisors and administrators and college teachers from August 12 to 16. This is the week immediately before the National Council meeting in nearby Northfield. The workshop wil have a staff including doctors Brownell, Dawson, Ruddell, and Sueltz. For information write: Director, Center for Continuation Study, University of Minnesota, Minneapolis 14, Minn.

BOOK REVIEW

Cross-Number Puzzles, Louis Grant Brandes, J. Weston Walch, Publisher, Box 1075, Portland, Maine, 1957. Paper, $8\frac{1}{2}$ by 11, Teacher Edition, 226 pages, $2.50, Student Edition, 156 pages, $2.00.

In *Cross-Number Puzzles,* Mr. Brandes suggests that here is "drill without Drudgery" and that this may be an answer to the charge that mathematics in both the elementary school and the high school is "the most hated subject." As the name implies, this is primarily a collection of cross-number puzzles in which the spaces are filled with numbers and operational symbols. Each puzzle has a heading suggesting the nature of the operations involved. These range from addition to powers and roots. The teacher edition gives the answers and a discussion of the significance and use of puzzles.

The author is not concerned with the use of this material as mere busywork, but rather he hopes that it will result in better learning of arithmetic. He cites some evidence that such has been the case. Certainly many teachers will find the puzzle interest stimulating to a whole class. Others may wish to use certain exercises for the more able pupil who needs a challenge and who has time for more work. Of course, a puzzle is not a basic teaching device and should not be used to replace the values to be gained from organized study of mathematics. It does provide review and practice within the limits of the content of the puzzle.

This reviewer believes that schools should have the Teacher Edition and then decide whether or not they regard this type of work of sufficient significance to warrant purchasing the Student Edition for class use. In general, the individual exercises are good, occasionally an exercise has little significance but provides a needed answer for the puzzle framework.

BEN A. SUELTZ

BOOK REVIEW

Printers' Arithmetic, F. C. Avis, Philosophical Library, Inc., 15 East Fortieth Street, New York 16, N. Y., 1956. Cloth, 145 pages, $4.75.

Here is a book intended to bring together the most important arithmetical problems confronting the printer in the composing room. It should be very useful in the printing trades and to teachers of printing in the schools. But it is also a valuable book for editors and others who plan publications. A teacher of mathematics will find many arithmetical problems which will be interesting to certain pupils.

The book must necessarily contain many technical terms which are used in the printing trades but these generally are defined and illustrated. One learns how to proceed from manuscript to the number of printed pages in terms of type size and page make-up. Various area problems including triangular and circular spaces are included. Fractions, particularly in an arrangement suggesting cancellation as a process, are numerous in various problems. Each different type of exercise is fully illustrated in calculation. The scope of the book is apparent when one realizes that more than thirty different types or kinds of problems are included.

This is not a manual for authors to use in the preparation of manuscript. The service of the book begins when a manuscript is ready for the printer.

BEN A. SUELTZ

Film Strip Review

Count 1, 2, 3, 4, 5, 21 Frames, black and white. The Filmstrip House, 347 Madison Ave., New York 17, N. Y.

This filmstrip shows collections of animals and objects in sequence of numbers from one to ten. The accompanying text asks the children to count the number of items in a scene. The arrangement of objects is satisfactory but it is a single arrangement for each number. In some frames there is a good deal of "fringe" material which aims to make the picture more interesting but by this same fact may be distracting to the essential count of the objects. This reviewer would like to see different arrangements of a given number of items, as for example, *five*. It would seem that real objects, rather than pictures, would serve better to use with children. These real things, whether they be pencils, books, or blocks, can be arranged and rearranged so that the "fiveness" of a number is evident for different configurations of the array.

KAREN SMITH

The Arithmetic Teacher

Items from the Annual Report of the Editor

The Growth of the Magazine

THE ARITHMETIC TEACHER was established in late 1953 as a quarterly journal of 32 pages each issue. The first issue appeared in February, 1954. The chart below indicates the progress that has been made in issues and circulation.

Year	*No. Issues*	*No. Pages*	*Circulation* (April Issue)
1954	4	128	3000 (+1000 "give away")
1955	5	170	4000
1956	6	256	5000
1957	6	312	6400

The growth in circulation has been in arithmetic progression, which is probably both fitting and healthy. If this growth can be maintained the editor will be happy. Probably, more promotional work should be done. The editor hesitates to advertise an item that is so closely associated with him. The growth is due largely to the efforts of many people in the National Council who have advertised the journal at many meetings.

Present Organization of the Magazine

Ben A. Sueltz of Cortland, New York serves as editor and John R. Clark of New Hope Pennsylvania and Mrs. Marguerite Brydegaard of San Diego, California serve as associate editors. Mr. Myrl Ahrendt, executive secretray of the National Council, serves as business manager. The business manager handles all business details including the advertising. The editor receives manuscripts from the field, solicits some to insure a wider coverage, and receives others from the associate editors who have stimulated their production. The associate editors are constantly looking for and encouraging the production of good manuscripts. All the editors work with authors to improve manuscripts.

It is the aim of the editors to produce a readable journal with articles that are clearly and comparatively simply written in language that is reasonably acceptable. The editor edits manuscripts, marks them for the printer, reads proof, "makes up" each issue, and does all the other little things involved in getting materials published. He assumes full responsibility for the journal in terms of writing, errors, topography, etc. For each article he writes an "Editor's Note," which aims to call attention to the main items in the article and to raise questions concerning the issues involved. The editor attempts to be judicious and objective in these notes. In a field like arithmetic, where there are differing points of view, it is probably desirable to continue these notes. Many people have expressed this. The Journal must not become a tool or propaganda medium for a selected point of view.

Range of Coverage and Authorship

THE ARITHMETIC TEACHER tries to cover the area of arithmetic with articles dealing with research, instruction, the curriculum, history, and new materials at all levels of the elementary school. Frequently an issue will be devoted largely to one area, as for example the March, 1957 issue deals mostly with the primary grades.

Beginning in September, 1957 the staff will be enlarged with someone in charge of a section that will deal with the topic "What is Happening in the Schools?" and another person handling book reviews. In this way the service of the journal will be enhanced.

During the first year of publication approximately 80% of the items printed were solicited. Now direct solicitation is down to perhaps 25% of the total. However, perhaps 50% of the items are stimulated and discussed with potential authors. *We need more articles from public school teachers and supervisors.*

In each issue of THE ARITHMETIC TEACHER we attempt to get a geographical range of authorship. To date we have had too few items from the Mid-South, the Southwest, and the Mountain states. It would be interesting to tabulate the number of articles that have come from the several states.